ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 18 | Information Theory and Coding
Homework 8 | November 16, 2010, SG1 – 15:15-17:00

Problem 1 (Binary Erasure Channel)**.** The binary erasure channel (BEC) is a channel with input $X \in \{0, 1\}$ and output $Y \in \{0, 1, e\}$ as shown in Figure 1. We will show that the capacity of the memoryless BEC is $C_{BEC} = 1 - \epsilon$.

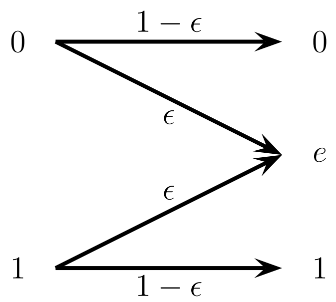

Figure 1: Binary Erasure Channel

(a) Let the input distribution be $\Pr\{X = 1\} = \alpha$ and $\Pr\{X = 0\} = 1 - \alpha$. Show that:

(i) $H(Y|X) = H_b(\epsilon)$

(ii) $H(Y) = H_b(\epsilon) + (1 - \epsilon)H_b(\alpha)$

(iii) $I_\alpha(X; Y) = (1 - \epsilon)H_b(\alpha)$

where $H_b(\zeta) = -\zeta \log \zeta - (1 - \zeta) \log(1 - \zeta)$.

(b) Show that $C_{BEC} = 1 - \epsilon$. What is the capacity achieving input distribution?

Problem 2 (Z Channel)**.** The Z channel is a channel with input $X \in \{0, 1\}$ and output $Y \in \{0, 1\}$ as shown in Figure 2. Here we will derive the capacity of the Z channel.

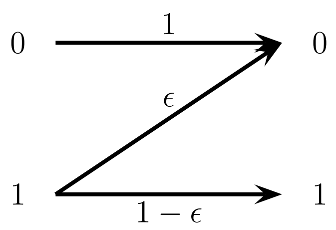

Figure 2: Z Channel

(a) Let the input distribution be $\Pr\{X = 1\} = \alpha$ and $\Pr\{X = 0\} = 1 - \alpha$. Show that:

(i) $H(Y|X) = \alpha H_b(\epsilon)$

(ii) $H(Y) = H_b(\alpha(1 - \epsilon))$

Therefore $I_\alpha(X; Y) = H_b(\alpha(1 - \epsilon)) - \alpha H_b(\epsilon)$.

(b) Show that $C_Z := \max_{0\le\alpha\le 1} I_\alpha(X;Y) = \log\left(1 + (1-\epsilon)\epsilon^{\frac{\epsilon}{1-\epsilon}}\right)$. What is the capacity achieving input distribution?

(c) Using MATLAB or any other tool, compute and compare the information rate $I_{1/2}(X;Y)$, achieved by the uniform input distribution, with the capacity C_Z for $0 \le \epsilon < 1$. Approximately how much do we lose (in percentage) on the information rate by using the uniform input distribution instead of the capacity achieving distribution?

Problem 3 (Symmetric Channels)**.** Consider a discrete memoryless channel with input $X \in \{1, 2, \cdots, m\}$ and output $Y \in \{1, 2, \cdots, n\}$. Let the channel transition probabilities be given by a matrix P where the entry in the xth row and yth column denotes the conditional probability $\Pr\{Y = y|X = x\}$. A channel is said to be *symmetric* if the rows of the matrix P are permutations of each other and the columns are permutations of each other. A channel is said to be *weakly symmetric* if the rows of the matrix P are permutations of each other and all the column sums $\sum_x p(y|x)$ are equal. Show that, for a weakly symmetric channel, the capacity C is

$$C = \log n - H(\text{row of transition matrix})$$

and this is achieved by a uniform distribution on the input alphabet. [Hint: Let $\mathbf{r}$ be a row of the transition matrix. Show that $I(X;Y) \le \log n - H(\mathbf{r})$. Use the condition for equality.]

Problem 4 (Fano Inequality)**.** Consider the following joint distribution on (X, Y):

$X \backslash Y$	a	b	c
1	1/6	1/12	1/12
2	1/12	1/6	1/12
3	1/12	1/12	1/6

Let $\hat{X}(Y)$ be an estimator of X based on Y and let $P_e = \Pr\{\hat{X}(Y) \neq X\}$.

(a) Find the estimator $\hat{X}(Y)$ that minimizes the probability of error P_e. What is the associated P_e?

(b) Evaluate Fano's inequality for this problem and compare with the answer from (a). Explain, if any, the cause for looseness of the bound.