ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 14 | Information Theory and Coding
Homework 7 | November 2, 2010, SG1 – 15:15-17:00

Problem 1. Suppose that a source X has alphabet $\mathcal{X}$ and it is known that its distribution is either $p_1(x)$ or $p_2(x)$, $\cdots$, or $p_K(x)$. Let $H_k = -\sum_x p_k(x)\log p_k(x)$ denote the entropy of the distribution $p_k, k = 1, \cdots , K$. Define $\hat{p}(x) = \max_{1\le k\le K} p_k(x)$, and $A = \sum_x \hat{p}(x)$.

(a) Show that $1 \le A \le K$.

(b) Show that there exists a prefix-free source code for X with codeword lengths $l(x) = \lceil -\log \hat{p}(x) + \log A\rceil$. [Hint: Use Kraft inequality]

(c) Show that, for a code as in (b), $\bar{L}_k = \sum_x p_k(x)l(x)$ (the average codeword length under distribution p_k) satisfies

$$H_k \le \bar{L}_k < H_k + \log A + 1.$$

(d) Suppose we have K binary Huffman codes for the alphabet $\mathcal{X}$ corresponding to the K different distributions $p_1, \cdots , p_K$. Let $l_k(x)$ be the length of the sequence that the k'th code assigns to the symbol $x \in \mathcal{X}$, $k = 1, \cdots , K$. If we do not know the exact distribution of X, describe a way to use these K Huffman codes to produce a prefix-free code which uses no more than

$$H(X) + \lceil\log_2 K\rceil + 1 \text{ bits/symbol.}$$

[Hint: For each symbol x, use the Huffman code which gives the shortest codeword.]

Problem 2. We showed in class that the maximum number of distinct words c into which a binary string of length n can be parsed satisfies

$$n > c\log_2\left(\frac{c}{8}\right).$$

Now derive a similar bound on n for the general case of a string over an alphabet of size K, that is, show that the maximum number of distinct K-ary words c into which a K-ary string of length n can be parsed satisfies

$$n > c\log_K\left(\frac{c}{K^3}\right)$$

Problem 3. The inequality in the above problem lower bounds n in terms of c. We will now show that n can also be upper bounded in terms of c.

(a) Show that, if $n \ge \frac{1}{2}m(m-1)$, then $c \ge m$. [Hint: Consider a string of $n = 0 + 1 + \cdots + (m-1)$ words.]

(b) Find a sequence for which the bound in (a) is met with equality.

(c) Now show that $n < \frac{1}{2}c(c+1)$.

Problem 4. Infimum of a subset S of real numbers is equal to the greatest real number (not necessarily in the subset) that is less than or equal to all elements of the subset. Supremum of a subset S of real numbers is equal to the smallest real number that is greater than or equal to all elements of the subset. For eg., if $S = \{x \in \mathcal{R} : 0 < x < 1\}$, $\inf(S) = 0$ and $\sup(S) = 1$ (note that $0, 1 \notin S$). The limit inferior of a sequence $\{a_n\}$ is defined by

$$\liminf_{n\to\infty} a_n := \lim_{n\to\infty}\left(\inf_{m\geq n} a_m\right).$$

The limit superior of a sequence is defined similarly with inf replaced by sup. Calculate the lim inf and lim sup of the following sequences:

(a)

$$a_n = (-1)^n \frac{(n+5)}{n}.$$

(b)

$$a_n = \begin{cases} \frac{1}{n} & \text{if } n = 3k \\ 1 - 2^{-n} & \text{if } n = 3k+1 \\ \left(1+\frac{1}{n}\right)^n & \text{if } n = 3k+2 \end{cases} \quad \text{for } k \in \{1, 2, \cdots\}.$$

Problem 5. Let X_i, $i = 1, 2, \cdots$, be an IID sequence with the following distribution:

$$X = \begin{cases} 1 & \text{with probability } 1/2 \\ 2 & \text{with probability } 1/4 \\ 3 & \text{with probability } 1/4. \end{cases}$$

If $Y_n = \left(\prod_{i=1}^n X_i\right)^{1/n}$, show that $Y \to 6^{1/4}$ as $n \to \infty$ with probability one.

Problem 6. Find the entropy rate associated with the random walk of a king on a 3×3 chessboard.

1	2	3
4	5	6
7	8	9

Find also the entropy rates of a rook and an even bishop. It is assumed that, at each step, a chess piece is moved to a different valid square than the current square and each of the possible valid squares is chosen with equal probability. [Hint: Entropy rates of a rook and an even bishop are easier to compute as they always have the same number of squares to move to from any valid position on the chessboard.]