ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 5 Information Theory and Coding
Homework 3 October 4th, 2010, SG1 – 15:15-17:00

Problem 1. Suppose that there are n coins which are either of the same weight or at most one coin has different weight than the others. The coins are to be weighted by a balance. The goal is to find out whether there exists a counterfeit coin and if so, is it heavier or lighter than the others.

a) For a given positive integer k, find an upper bound on the number of coins, n, for which k usage of the balance is sufficient.

b) Demonstrate a strategy for $n = 12$ and $k = 3$.

Problem 2. A biased coin is flipped until both head and tail show up at least once, Suppose that the probability that the outcome of the coin flip is head is p. Let X denote the number of flip required. Find the entropy $H(X)$ in bits.

Problem 3. Suppose that $\mathcal{A}$ is a finite set and P is a probability mass function defined on the set $\mathcal{A}$. Let X be a random variable which takes its value from the set $\mathcal{A}$ with respect to P. Let $Y = f(X)$ where f is some function.

a) Show that for every function f we have $H(X) \geq H(Y)$. Give both intuitive and precise mathematical proof.

b) Prove that $H(X) = H(Y)$ if and only if f is a one-to-one function.

Problem 4. Consider the following code generating for a random variable X which takes on m values $\{1, 2, \ldots, m\}$ with probabilities $p_1 \geq p_2 \geq \ldots \geq p_m$. Define $S_i = \sum_{j=1}^{i-1} p_j$. Then the codeword for i is the number $S_i \in [0, 1]$ rounded off to $\log\lceil \frac{1}{p_i} \rceil$ bits (i.e. binary representation of S_i). Show that the code constructed by this process is prefix-free and the average length of the code satisfies: $H(X) \leq L < H(X) + 1$.

Problem 5. Find the (a) binary (b) ternary Huffman code for the random variable X with the probability mass function $p = \big(1/12, 1/12, 1/8, 1/8, 1/4, 1/3\big)$.