ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 23 Signal Processing for Communications
Homework 9 May 2, 2011, INF 1 - 10:15-12:00

Problem 1 (DFT Revisit). (i) Let $x[n]$ be a real N-point sequence and $X[k]$ be its N-point DFT. Let $x_1[n]$ be a sequence such that $X_1[k] = \text{real}\{X[k](-1)^k\}$. Given that N is even, find $x_1[n]$ in terms of $x[n]$.

(ii) Let $x_1[n]$ and $x_2[n]$ be two real N-point sequences such that $x_1[n]$ is symmetric, and $x_2[n]$ is anti-symmetric. Let $X_1[k]$ and $X_2[k]$ denote their corresponding N-point DFTs. Given $y[n] = x_1[n] + x_2[n]$ with its DFT denoted as $Y[k]$, explain how $X_1[k]$ and $X_2[k]$ can be recovered from $Y[k]$.

Problem 2 (Limits of Z-transform). Let $X(z)$ be z-transform of a causal discrete signal $x[n]$, compute the following equations in terms of $x[n]$:

(i) $X(1)$. What does it show?

(ii) $\lim_{z\to\infty} X(z)$.

(iii) $\lim_{z\to\infty} z(X(z) - x[0])$.

(iv) $-z\frac{\mathrm{d}X(z)}{\mathrm{d}z}$.

(v) $\lim_{z\to\infty} -z^2\frac{\mathrm{d}X(z)}{\mathrm{d}z}$.

Problem 3 (Stochastic Processes). Consider a discrete random process $X[n] = \sin(\omega n+\theta)$ such that θ is a random variable with uniform distribution on $[0, 2\pi]$ and $\omega \in \mathbb{R}$.

(i) Find the mean and autocorrelation function of $X[n]$. Is it a wide-sense stationary process?

Define
$$Y[n] = X[n] + \beta X[n-1],$$
where $\beta \in \mathbb{R}$.

(ii) Compute the power spectral density $P_Y(e^{j2\pi f})$.

Now assume that $X[n]$ is a zero-mean wide-sense stationary process with autocorrelation function given by
$$R_X[k] = \sigma^2\alpha^{|k|},$$
for $|\alpha| < 1$.

(iii) Compute the power spectral density $P_Y(e^{j2\pi f})$.

(iv) For which values of β does $Y[n]$ corresponds to a white noise?

Problem 4 (Min. Mean Squared Error Estimator*)**.** In this problem, we want to approximate random variable X in terms of a given set of observations $\{Y_1, \cdots, Y_l\}$ such that Y_i, for $1 \leq i \leq l$, is a random variable correlated to X.

Consider the case that we have only one observation, Y. Define $\hat{X} = h(Y)$ as an estimator of X. Then, we estimate the value of X by knowing the value of observation Y. An estimator is called **unbiased** estimator, if $\mathbb{E}(X) = \mathbb{E}(\hat{X})$. Assume that X and Y are continuous random variables with the joint probability density function (PDF) $P_{X,Y}(x, y)$. The marginal PDF of X and Y are denoted by $P_X(x)$ and $P_Y(y)$.

Let $\mathcal{H}$ be a Hilbert space of random variables with an inner product defined by

$$\langle U, V \rangle = \mathbb{E}(UV^*) = \int uv^* P_{X,Y}(x, y)\mathrm{d}x\mathrm{d}y, \tag{1}$$

where $U, V \in \mathcal{H}$. the space $\mathcal{H}$ contains X and Y and all the random variables $f(X, Y)$ such that $f(\cdot)$ is a continuous function. Moreover, the subspace of random variable Y contains random variables Y and $f(Y)$ for all continuous function $f(\cdot)$.

i) Prove that $\langle U, V \rangle$ in (1) is an inner product?

Define mean squared error as

$$\langle x - \hat{x}, x - \hat{x} \rangle = \mathbb{E}(|x - \hat{x}|^2) = \int |x - \hat{x}|^2 P_{X,Y}(x, y)\mathrm{d}x\mathrm{d}y.$$

ii) Among the linear unbiased estimators, i.e. $\hat{X} = aY + b$, find the estimator with the minimum mean squared error.

iii) Prove that $\hat{X} = h(Y) = \mathbb{E}(X \mid Y)$ has the minimum mean squared error among all unbiased estimators.

Hint 1: Use projection theorem.

Hint 2: Due to definition of conditional expectation, if $Z = \mathbb{E}(X \mid Y)$ then $\mathbb{E}((X - Z)f(Y)) = 0$ for all continuous function of $f(\cdot)$.

$(*)$ Just for fun. Such problems are out of focus of this course :-).