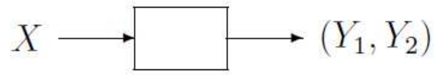
Problem 1. A binary input channel C is given. Let c be the capacity of C. Suppose that Y_1 and Y_2 are two conditionally independent observations of the channel, i.e., $p(Y_1, Y_2 \mid X) = p(Y_1 \mid X)p(Y_2 \mid X)$. This situation is depicted in the following figure. Denote the resulting channel by $C^{(*2)}(C)$.



Prove the following:

(i) $I(Y_1; Y_2) + I(X; Y_2) = 2I(Y_1; X) - I(Y_1; X|Y_2)$.

(ii) Show that the capacity of $C^{(*2)}(C)$ is at most $2c$.

Problem 2. Consider again the setup as in Problem 1 but where the underlying channel C is now a binary erasure channels (BEC) with erasure probability p, call it BEC(p).

(i) What is the capacity of BEC(p)?

(ii) What is the capacity of $C^{(*2)}$(BEC(p)) in terms of the capacity of BEC(p)?

(iii) Bonus: Can you also find the capacity of this setup, but now assuming that the underlying channel is a binary symmetric channel (BSC) with error probability p? I.e., what is the capacity of $C^{(*2)}$(BSC(p))?

Remark: It can be shown that for any binary-input memoryless symmetric channel C of capacity c the resulting capacity $C^{(*2)}(C)$ is always upper bounded by $C^{(*2)}$(BEC(p)) (where p is chosen so that the capacity of BEC(p) is equal to c) and lower bounded by $C^{(*2)}$(BSC(p)) (where p is chosen so that the capacity of BSC(p) is equal to c). This result is called *extremes of information combining* and will play an important role when we talk about *polar codes.*

Problem 3. Let X be a continuous random variable with density function f and let $g_n = \frac{1}{n} f(nX)$ where n is some positive integer number.

(i) Find the differential entropy $h(X)$, if $f(x) = \lambda e^{-\lambda x}, x \geq 0$.

(ii) Prove that g_n is also a density function.

(iii) Suppose that Y_n is a continuous random variable with density function g_n. What is the relationship between $h(X)$ and $h(Y_n)$?

Problem 4. Consider two discrete memoryless channels $(\mathcal{X}_1; p(y_1|x_1); \mathcal{Y}_1)$ and $(\mathcal{X}_2; p(y_2|x_2); \mathcal{Y}_2)$ with capacities C_1, and C_2 respectively. A new channel $(\mathcal{X}_1 \times \mathcal{X}_2; p(y_1|x_1) \times p(y_2|x_2); \mathcal{Y}_1 \times \mathcal{Y}_2)$ is formed where the input is (x_1, x_2), $(x1 \in \mathcal{X}_1$ and $x_2 \in \mathcal{X}_2$, and where the output is (y_1, y_2). Find the capacity of this channel.