ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 3 — Signal Processing for Communications

Homework 2 — February 28, 2011, INF 213 - 10:15-12:00

Problem 1 (Any Basis of a Hilbert Space has Same Cardinality)**.** Let H be a Hilbert space. We say that $B = \{x_i\}$ forms a basis for H if B is a maximal orthonormal family in H, i.e., the elements x_i are orthonormal and there exists no element $x \in H$ which is orthogonal to all $\{x_i\}$.

Recall that in the space $\mathbb{C}^n$ any set of orthogonal vectors has cardinality at most n.

Consider a Hilbert space H and assume that $B = \{x_1, \dots, x_n\}$ as well $B' = \{x'_1, \dots, x'_m\}$ form bases for H. Show that $n = m$.

Hint: Write $x'_i = \sum_{j=1}^n \alpha_{ij} x_j$ and now look at $\langle x'_k, x'_l \rangle$.

Problem 2 (Gram-Schmidt)**.** Consider the Hilbert space $\mathbb{R}^4$. Apply the Gram-Schmidt procedure to the subspace spanned by the set of the following three vectors:

$$u_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, u_2 = \begin{pmatrix} 5 \\ 1 \\ 1 \\ 1 \end{pmatrix}, u_3 = \begin{pmatrix} -3 \\ -3 \\ 1 \\ -3 \end{pmatrix}$$

Gram-Schmidt Orthonormalization Procedure: Given $\{x_n\}_{n\in\mathbb{N}}$ in a Hilbert space H, an orthonormal system $\{e_n\}_{n\in\mathbb{N}}$ can be generated by applying:

$$e_0 = \frac{x_0}{||x_0||}$$
$$e_{n+1} = \frac{x_{n+1} - \sum_{k=1}^n < x_{n+1}, e_k > e_k}{||x_{n+1} - \sum_{k=1}^n < x_{n+1}, e_k > e_k||}$$

Problem 3 (Various Norms)**.** Consider the vector space $\mathbb{C}^N$, i.e., the space of complex N-tuples, $x = [x_1, x_1, \cdots, x_N]$. Prove that both v_1 as well as v_2 are norms on $\mathbb{C}^N$, where

$$v_1(x) = \sum_{k=1}^N |x_k|, \quad v_2(x) = \left(\sum_{k=1}^N |x_k|^2 \right)^{\frac{1}{2}}.$$

Problem 4 (Convergent Sequences are Cauchy Sequences)**.** Show that in any metric space any convergent sequence is a Cauchy sequence.

Problem 5 (Incompleteness of $\mathbb{Q}$)**.** Consider the space of $\mathbb{R}$ with the metric $d(x, y) = |x - y|$ for $x, y \in \mathbb{R}$. We have seen in class that this space is complete.

(i) Let a_n be a sequence recursively defined by $a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n}$, $a_1 = 2$. Show that a_n is a bounded, decreasing sequence of rational numbers. Prove that this sequence converges. What is its limit?

(ii) By using part (i), prove that the space of $\mathbb{Q}$, rational numbers, with the metric of absolute value is not complete.

Problem 6 (Properties of DFT)**.** Let $x[n]$ and $y[n]$ be two finite-length sequences of length N. Let $X[k]$ and $Y[k]$ be their corresponding N-point DFTs. Prove the following properties of the DFT.

1) Linearity: $\forall \alpha, \beta \in \mathbb{C}$

$$\alpha x[n] + \beta y[n] \xrightarrow{\text{DFT}} \alpha X[k] + \beta Y[k].$$

2) Circular Shift:

$$x[(n-m) \mod N] \xrightarrow{\text{DFT}} e^{-j(\frac{2\pi}{N})km} X[k].$$

3) Duality:

$$X[n] \xrightarrow{\text{DFT}} Nx[(-k) \mod N].$$

4) Symmetries:

$$\begin{aligned} x^*[n] &\xrightarrow{\text{DFT}} X^*[(-k) \mod N]. \\ x_{ep}[n] = \tfrac{1}{2}\{x[n] + x^*[(-n) \mod N]\} &\xrightarrow{\text{DFT}} \mathrm{Re}\{X[k]\}. \\ x_{op}[n] = \tfrac{1}{2}\{x[n] - x^*[(-n) \mod N]\} &\xrightarrow{\text{DFT}} j\,\mathrm{Im}\{X[k]\}. \end{aligned}$$

Note: $x_{ep}[n]$ and $x_{op}[n]$ are called the periodic conjugate-symmetric and periodic conjugate-antisymmetric parts of $x[n]$, though they are not periodic sequences.

5) Cyclic Convolution:

$$\sum_{m=0}^{N-1} x[m]y[(n-m) \mod N] \xrightarrow{\text{DFT}} X[k]Y[k].$$