ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 26 Information Theory and Coding
Homework 12 (Graded DUE 21-12-2010) December 14, 2010, SG1 – 15:15-17:00

The objective of this homework is to get familiar with the practical issues of polar codes. The channel which we consider throughout the homework is the binary erasure channel with erasure probability equal to 0.5, which we denote by BEC(0.5). The homework consists of three parts: analysis, encoding and decoding. In each part you need to write a small computer program to answer the question. You can use any programming language you want (C, C++, Matlab, Mathematica). You can work in groups of up to 4 people. Only one report by each group is needed. Mark each group clearly.

Problem 1 (Analysis). (a) As remarked above the channel we consider is the BEC(0.5). Denote the block-length by $N = 2^n$ and the rate by R. Recall that the sum of the smallest NR Bhattacharyya parameters of the sub-channels $\{W_N^{(i)}\}_{1 \leq i \leq N}$ is an upper bound on the block error probability using polar codes with an successive cancellation (SC) decoder. For the following four values of n, namely $n = 10, 15, 20$ and 25, plot the this upper bound (sum of the smallest NR Bhattacharyya values) in terms of R for $0 < R < 0.5$ in a logarithmic scale. Plot all these four curves in one figure.

(b) Let $R = 0.25$ and plot the sum of the smallest NR Bhattacharyya parameters of the sub-channels $\{W_N^{(i)}\}_{1 \leq i \leq N}$ in terms of n for $0 \leq n \leq 25$. Also, plot the value of the function $\log(|\log(.)|)$ applied on the sum of least NR Bhattacharyya values in terms of n for $0 \leq n \leq 25$. Deduce from this plot that as n grows large, the best achievable block error probability using polar codes and the SC decoder is at most of order $o(2^{-N^\beta})$ for some value of $\beta > 0$. Can you predict the value of β from the plot?

Problem 2 (Encoding). (a) If we want to use polar codes to transmit data on the channel BEC(0.5) with $R = 0.25$ and $N = 2^{10}$, what are the indices of the good sub-channels? Put them in a text file called *indices.txt*.

(b) Pick the first letter of the surname of each member of your group. Arrange these letters in any order. What is the resulting sequence of letters?

(c) Assume we convert the 26 letters of the English alphabet to 5-bit binary strings in the natural way such that 'A' is converted to 00000 and 'Z' is converted to 11001. Convert the letters you picked in part (b) to 5-bit binary strings and concatenate the strings with the same order as part (b) to form a new binary string. Add a sufficient number of '1's to the end of this new string so that its length reaches 256. Encode the final string using a polar code with $R = 0.25$ and $N = 2^{10}$ (you have found the good indices in part (a)). Put the encoded data in a file called *encoded.txt*.

Problem 3 (Decoding). The objective of this problem is to illustrate a way to decode polar codes when the channel is the BEC.

Assume that we have a linear code with a $k \times N$ generator matrix G. We denote the data to be encoded by a k dimensional row vector u. So at the transmitter the N dimensional row vector $x = uG$ is sent through the channel BEC(ϵ) which erases each element of x with probability ϵ independently. Now let I be the (ordered) set of indices of the non-erased elements. Denote by x_I the part of x with indices in I. Also denote by G_I the part of G consisting of the rows of G with indices in I.

(a) Use the identity $x = uG$ to show that optimal (MAP) decoding can be done by solving the system of linear equations $x_I = uG_I$.

(b) Prove that if $\text{rank}(G_I) < |I|$, then we have an error in the optimal (MAP) decoding procedure.

(c) Assuming the settings of Problem 2, transmit the encoded string you found in Problem 2 through the channel BEC(0.5) and decode the received vector using the optimal algorithm given in part (a). Do this experiment 1000 times and define the average probability of error p_e to be the number of times that you have an error in your decoding procedure divided by 1000. What value of p_e do you get?

(d) (Bonus, extra 25%) Explain how can you do the successive cancellation (SC) decoding procedure that was taught in class assuming the channel is a BEC. In particular explain how can you decode the information bit u_i having the knowledge of all the previous bits $u_1, \cdots, u_{i-1}$. When do you have an error? Do the same experiment as part (c) but now using the SC decoder. What do you get for p_e? (HINT: Like in part (a) you have to solve a system of linear equations. Note that the inverse of the matrix $G_n = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]^{\otimes n}$ is itself, i.e., $G_n = G_n^{-1}$ (you can prove this by induction). Try to start with $x = uG_n$, where u is the N-dimensional vector consisting of both the information and fixed bits (which we fix them to zero). Then deduce that $u = xG_n$ and continue with considering the set of erased and non-erased bits in the output).