Handout 22 Information Theory and Coding
Homework 10 (Graded) DUE 7-12-2010 November 30, 2010, SG1 – 15:15-17:00

Problem 1. Consider the channel C depicted in Figure 1 (a). In this channel C, the input X is taken from the set of real numbers and $Z \sim \mathcal{N}(0,1)$ is an additive random Gaussian noise with distribution $\mathcal{N}(0,1)$ which is independent from X. The observation Y is equal to $X+Z$. Notice that Y is a real valued random variable. We learned that the capacity of this channel with power constraint P is equal to $\frac{1}{2}\log(1+P)$.

In practice it is not convenient to have to deal with a continuous input alphabet. But, as we will see, if we use a suitable discrete input alphabet (rather than the capacity achieving Gaussian distribution) then we can achieve rates arbitrarily close to the optimal setup.

(i) Consider the following channel C_2 in which $X \in \{-1,1\}$, $Z \sim \mathcal{N}(0,1)$ is a Gaussian noise and Y_2 is defined as $Y_2 = X + Z$. More precisely, we are only allowing binary inputs. Show that the resulting channel is symmetric and so the optimum input distribution is the uniform one. There is no closed-form solution for the capacity of this channel. But you can write it down as an integral. State this integral and use Matlab (or Mathematica) to plot the capacity as a function of P. Compare this to the capacity of the channel C. Any comments?

(ii) So far we have quantized the input. Sometimes it is also convenient to quantize the output. What is the capacity of the channel C_2' defined as following:

$$Y_2' = \begin{cases} 1 & \text{if } Y_2 > 0, \\ -1 & \text{if } Y_2 \leq 0 \end{cases}$$

Plot this capacity also on the previous plot and compare.

Problem 2. Let $Y = X+Z$, where X and Z are independent, zero mean random variables with variances a and b, respectively. In this problem we want to prove the following inequality : $h(X|Y) \leq \frac{1}{2}\log(\frac{2\pi eab}{a+b})$.

(i) Prove that $h(X|Y) \leq \frac{1}{2}\log(2\pi e E_Y(\mathrm{Var}(X|Y)))$.

(ii) Show that $E_Y(\mathrm{Var}(X|Y)) \leq \frac{ab}{a+b}$.

(iii) Conclude that $h(X|Y) \leq \frac{1}{2}\log(\frac{2\pi eab}{a+b})$.

Hint: For part (ii) you can use the following facts

- $E_X((E(X|Y)-X)^2) = Var(X|Y)$. (Consider both sides as functions of Y.)
- In this problem, the minimum of $E((X-\hat{X})^2)$ over all estimator for X in terms of the observation Y (i.e the minimum over all the functions of Y) is obtained when $\hat{X}(y) = E(X|Y=y)$.

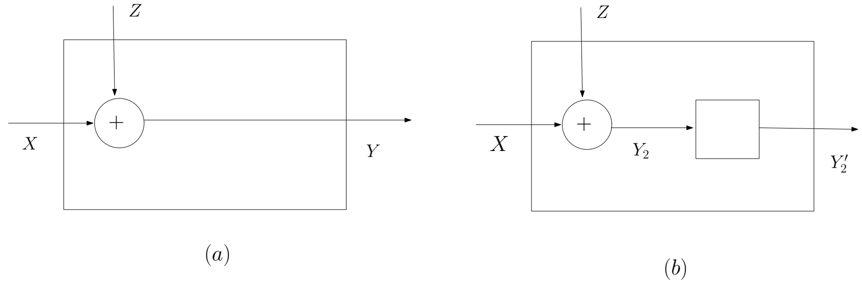

Figure 1: Problem 1 – (a): Channel C and (b): Channel C_2'

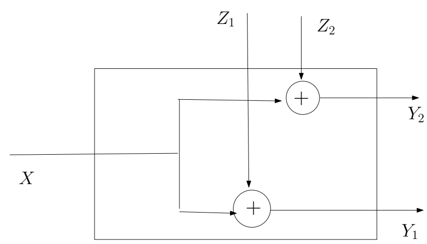

Figure 2: Problem 3

Problem 3. Consider the channel with input X and output (Y_1, Y_2) where $Y_1 = X + Z_1$ and $Y_2 = X + Z_2$ in which Z_1 and Z_2 are two Gaussian additive random noises both independent from X. Suppose that $(Z_1, Z_2) \sim \mathcal{N}(0, K)$ where K is the covariance matrix of Z_1 and Z_2. Let P be the power constrain of the input. (See Figure 2) Compute the capacity of the channel if

$$K = \begin{bmatrix} M & M\sigma \\ M\sigma & M \end{bmatrix}$$

and

(i) $\sigma = 1$

(ii) $\sigma = \frac{1}{2}$

(iii) $\sigma = -1$

Problem 4. Consider a vector Gaussian channel described as follows:

$$Y_1 = X + Z_1$$
$$Y_2 = Z_2$$

in which $X \in \mathbb{R}$ is the input to the channel constrained in power to P; Z_1 and Z_2 are jointly Gaussian random variables with $E(Z_1) = E(Z_2) = 0, E(Z_1^2) = \sigma_1^2, E(Z_2^2) = \sigma_2^2$ and $E(Z_1 Z_2) = \sigma_3^2$, with $\sigma_3 \leq \sigma_1, \sigma_2$, and independent of the channel input.

(i) What is the capacity of the channel 1 depicted in Figure 4, (a)?

(ii) What is the capacity of the channel 2 depicted in Figure 4 (b)?

(iii) What is the capacity of the channel 3 depicted in Figure 4 (c)?

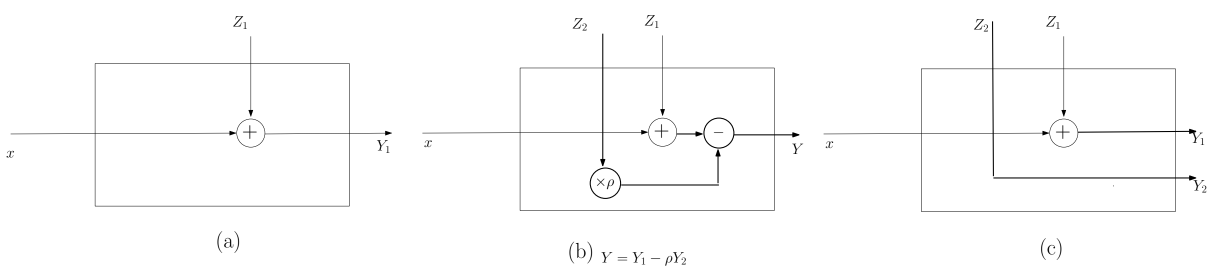

Figure 3: Problem 4, (a): Channel 1, (b): Channel 2, (c): Channel 3.