ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 14 Advanced Digital Communications
Midterm November 15, 2010

3 problems, 89 points
3 hours
2 sheets of notes allowed
Good Luck!

PROBLEM 1. (29 points) A complex number x is observed K times, the observations $Y_1, \ldots, Y_K$ given by

$$Y_k = h_k x + Z_k, \quad k = 1, \ldots, K$$

where $h_1, \ldots, h_K$ are fixed and known complex numbers and Z_k are independent, circularly symmetric Gaussian random variables with $E\big[|Z_k|^2\big] = \sigma_k^2$.

(a) (5 pt) Suppose the receiver combines all the K complex-valued observations into a scalar decision variable V by forming the linear combination

$$V = \sum_k g_k^* Y_k = \langle g, Y \rangle.$$

Express V in the form $V = \gamma x + Z$ and find γ, $E[Z]$ and $E\big[|Z|^2\big]$.

(b) (7 pts) Show that the 'gain-to-noise ratio' of V satisfies

$$\frac{|\gamma|^2}{E\big[|Z|^2\big]} \le \sum_k \frac{|h_k|^2}{\sigma_k^2}$$

(c) (5 pts) Show that the choice $g_k = h_k/\sigma_k^2$ reaches the upper bound above.

(d) (7 pts) With g_k as in part (c), show that $p(y_1, \ldots, y_K|x)$ can be factorized as

$$p(y_1, \ldots, y_K|x) = a(y_1, \ldots, y_K)\, b\big(\mathrm{Re}\{x^* v\}\big)\, c(x)$$

where $v = \langle g, y \rangle$, and $a(\cdot)$, $b(\cdot)$, $c(\cdot)$ are functions only of the indicated variables in their arguments.

(e) (5 pts) Again with g_k as in part (c), is V a sufficient statistic to estimate x? Explain.

PROBLEM 2. (36 points) We have a communication system where a sequence x_k is transmitted in the form

$$x(t) = \sum_k x_k \phi(t - kT).$$

The x_k's each belong to the two point constellation $\{+\sqrt{A}, -\sqrt{A}\}$. The pulse shape ϕ has Fourier transform given by

$$\Phi(f) = \begin{cases} \sqrt{2T_0}\cos(\pi f T_0) & |f|T_0 < 1/2 \\ 0 & \text{else.} \end{cases}$$

(a) (5 pts) Sketch $|\Phi(f)|^2$. Hint: $\cos^2(\theta) = (\cos(2\theta) + 1)/2$.

(b) (5 pts) What is the smallest value of T so that the collection $\phi_k(t) = \phi(t - kT)$, $k \in \mathbb{Z}$ is an orthonormal collection?

(c) (7 pts) Suppose T is chosen as above, and that the received signal is given by

$$y(t) = x(t) - 2x(t - T) + z(t)$$

where $z(t)$ is white Gaussian noise with spectral density $N_0/2$.

Denoting the matched filter output samples y_k as

$$y_k = (q * x)_k + z_k,$$

find the filter coefficients q_k and the noise spectra $S(D)$.

(d) (7 pts) Design a stable and causal whitening filter so that the composition of the whitening filter and the filter q is also causal and stable.

(e) (5 pts) Denoting the output of the whitening filter as

$$\tilde{y}_k = (h * x)_k + \tilde{z}_k$$

Find the coefficients h_k and $E[|\tilde{z}_k|^2]$.

(f) (7 pts) Suppose the receiver is provided with the true values of the transmitted symbols up to (and including) time $k - 1$, and is asked to estimate x_k from y_k. Find the error probability of the receiver's decision as a function of A/N_0.

Problem 3. (24 points) Consider a communication system where the transmitted waveform is

$$x(t) = \sum_k x_k \phi(t - 2k)$$

where $\phi(t) = \text{sinc}(t)$. (Note that the signaling interval T equals 2.)

(a) (5 pts) Does the collection $\phi_k(t) = \phi(t - 2k)$ form a set of orthonormal signals?

(b) (5 pts) The signal $x(t)$ is transmitted over the channel with impulse response $g_c(t) = \delta(t - 1/2) + \delta(t + 1/2)$. Let the output of the filter be denoted

$$\sum_k x_k \psi(t - 2k).$$

Find the Fourier transform $\Psi(f)$ of $\psi(t)$. Do the $\psi(t - 2k)$'s form an orthogonal collection?

(c) (7 pts) The receiver filters the received signal through the matched filter $\psi^*(-t)$, and samples the output of the matched filter at instants $t = 2k + \delta$ where δ is a small timing offset with $|\delta| \ll 1$. Denoting the (noise-free) matched filter output y_k as

$$y_k = (q * x)_k$$

find q_k in terms of δ. Hint: $u(t) = \text{sinc}(t)/(1 - t^2)$ has Fourier transform

$$U(f) = \begin{cases} 2\cos^2(\pi f) & |f| < 1/2 \\ 0 & \text{else.} \end{cases}$$

(d) (7 pts) Find positive constants α and β so that

$$|q_k| \le \alpha \frac{\sin(\pi|\delta|)}{|k|^\beta}$$

and upper bound the energy of the intersymbol-interference term

$$\sum_{\ell \ne 0} q_\ell x_{k-\ell}$$

that corrupt y_k. (You may assume that x_k is an i.i.d. sequence with $E[|x_0|^2] = \mathcal{E}$.)