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Handout 16 Advanced Digital Communications
Midterm Solutions November 24, 2010

PROBLEM 1.

(a) As $Y_k = h_k x + Z_k$, $V = \langle g, Y\rangle = \sum_k g_k^* Y_k$ equals $\gamma x + Z$ with

$$\gamma = \sum_k g_k^* h_k = \langle g, h\rangle, \quad \text{and} \quad Z = \sum_k g_k^* Z_k.$$

Since Z_k's are independent circularly symmetric Gaussians, Z, being a linear combination of them, is also Gaussian and circularly symmetric. In particular, $E[Z] = 0$, and

$$E\big[|Z|^2\big] = \sum_k |g_k|^2 E\big[|Z_k|^2\big] = \sum_k |g_k|^2 \sigma_k^2.$$

(b&c) By the Cauchy-Schwartz inequality $|\langle a, b\rangle|^2 \le \langle a, a\rangle\langle b, b\rangle$. We thus have

$$|\gamma|^2 = \Big|\sum_k g_k^* h_k\Big|^2 = \Big|\sum_k (g_k\sigma_k)^*(h_k/\sigma_k)\Big|^2 \le \Big(\sum_k |g_k|^2\sigma_k^2\Big)\Big(\sum_k |h_k|^2/\sigma_k^2\Big)$$

and thus the 'gain-to-noise ratio' satisfies

$$\frac{|\gamma|^2}{E\big[|Z|^2\big]} \le \sum_k \frac{|h_k|^2}{\sigma_k^2}.$$

The equality in Cauchy-Schwartz inequality holds when $a = b$; in this case when $g_k\sigma_k = h_k/\sigma_k$.

(d) Conditional on x, the random variables Y_k are independent, and since $Y_k = h_k x + Z_k$,

$$p(y_k|x) = \frac{1}{\pi\sigma_k^2}\exp\Big(-\frac{1}{\sigma_k^2}|y_k - h_k x|^2\Big)$$

As $|y_k - h_k x| = |y_k|^2 - 2\,\mathrm{Re}\{x^* h_k^* y_k\} + |h_k x|^2$, we see that

$$p(y_1, \dots, y_K|x) = \frac{1}{\prod_k(\pi\sigma_k^2)}\exp\Big(-\sum_k \frac{|y_k|^2}{\sigma_k^2}\Big)$$
$$\times \exp\Big(2\,\mathrm{Re}\Big\{x^*\sum_k \frac{h_k^* y_k}{\sigma_k^2}\Big\}\Big) \times \exp\Big(-\sum_k \frac{|x h_k|^2}{\sigma_k^2}\Big)$$

which is in the required form.

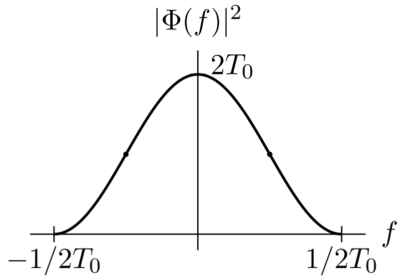
(e) $V = \sum_k h_k^* Y_k/\sigma_k^2$ is a sufficient statistic to estimate x. To show this we need to demonstrate that $p(x|y_1, \dots, y_K)$ is a function of x and v only. But

$$\begin{aligned}
p(x|y) &= \frac{p(y|x)p(x)}{\int p(y|\tilde{x})p(\tilde{x})\,d\tilde{x}} && \text{(Bayes rule)}\\
&= \frac{a(y)b(\mathrm{Re}\{x^*v\})c(x)p(x)}{a(y)\int b(\mathrm{Re}\{\tilde{x}^*v\})c(\tilde{x})p(\tilde{x})\,d\tilde{x}} && \text{(part (d))}\\
&= \frac{b(\mathrm{Re}\{x^*v\})c(x)p(x)}{\int b(\mathrm{Re}\{\tilde{x}^*v\})c(\tilde{x})p(\tilde{x})\,d\tilde{x}} && \text{as required.}
\end{aligned}$$

Problem 2.

(a)

$$|\Phi(f)|^2 = \begin{cases} 2T_0\cos^2(\pi f T_0) & |f|T_0 < 1/2 \\ 0 & \text{else.} \end{cases}$$
$$= \begin{cases} T_0(1+\cos(2\pi f T_0) & |f|T_0 < 1/2 \\ 0 & \text{else.} \end{cases}$$



Note, in particular, the symmetry around the point $(1/(4T_0), T_0)$.

(b) The shifts of $|\Phi(f)|^2$ by multiples of $1/(2T_0)$ sums to $2T_0$. Thus whenever T is an integer multiple of $2T_0$ we get an orthonormal collection of ϕ_k's. The smallest value of T is thus $2T_0$.

(c) As $y(t) = \sum_k x_k \psi(t-kT) + z(t)$ with $\psi(t) = \phi(t) - 2\phi(t-T)$, the matched filter output is

$$y_k = (q * x)_k + z_k$$

with $q_k = \langle \psi(t), \psi(t-kT) \rangle$ and the noise spectra given by $Q(D)N_0/2$. The inner product in the expression of q_k evaluates to

$$\begin{aligned} q_k &= \langle \phi(t) - 2\phi(t-T), \phi(t-kT) - 2\phi(t-T-kT) \rangle \\ &= \delta_k - 2\delta_{k+1} - 2\delta_{k-1} + 4\delta_k \\ &= \begin{cases} 5 & k = 0 \\ -2 & k = -1, 1 \\ 0 & \text{else} \end{cases} \end{aligned}$$

and thus $Q(D) = -2D^{-1} + 5 - 2D$.

(d) Observe that $Q(D) = 4(1 - D/2)(1 - 1/(2D))$. Choosing $W(D) = \frac{-D/2}{1-D/2}$ will ensure that the poles of $W(D)$ are outside the unit circle (hence $W(D)$ is causal and stable) and make $Q(D)W(D) = 1 - 2D$ leaving a causal and stable filter. The factor D on the numerator of $W(D)$ has no effect on the whiteness of the output noise since it signifies only a delay by one time unit.

(e) With the choice above $H(D) = Q(D)W(D) = 1 - 2D$; so $h_0 = 1$, $h_1 = -2$, and all other h_k equal zero. The noise $\tilde{z}_k$ is white with spectra $N_0/2Q(D)W(D)W(1/D^*)^* = N_0/2$, so $\tilde{z}_k$ has variance $N_0/2$.

(f) As $y_k = x_k - 2x_{k-1} + \tilde{z}_k$, and since the receiver is provided with the value of x_{k-1}, the receiver can compute $y_k + 2x_{k-1} = x_k + \tilde{z}_k$ from which to estimate x_k. As x_k belongs to the binary constellation $\pm\sqrt{A}$, the probability of an incorrect decision is the probability that a Gaussian noise of variance $N_0/2$ exceeds $\sqrt{A}$, which is $Q(\sqrt{2A/N_0})$. (The problem does not specify if the original noise $z(t)$ is real or complex, the computation here assumes real $z(t)$.)

Problem 3.

(a) The Fourier transform of $\mathrm{sinc}(t)$ is $\mathrm{rect}(f)$ — the function taking the value 1 or 0 according to $|f| < 1/2$ or not. As $|\mathrm{rect}(f)|^2 = \mathrm{rect}(f)$, its shifts by integers sum to 1. Thus, we see that the collection $\{\mathrm{sinc}(t-k)\}$ forms an orthonormal set. In particular, the sub-collection $\{\mathrm{sinc}(t-2k)\}$ also forms an orthonormal collection.

(b) As $\psi(t) = \phi(t-1/2)+\phi(t+1/2) = \phi(t)*[\delta(t-1/2)+\delta(t+1/2)]$, its Fourier transform is $\mathrm{rect}(f)2\cos(\pi f)$. Consequently $|\Psi(f)|^2 = 4\cos^2(\pi f)$ on the interval $|f| < 1/2$ and zero elsewhere, and has the same shape as in the figure in problem 2a. Its shifts by integer multiples of 1/2 thus sum to 2, which means that the collection $\psi(t-2k)$ is an orthogonal collection. (They are not orthonormal, but would have been if they were scaled by $1/\sqrt{2}$.)

(c) As $y(t) = \sum_\ell x_\ell \psi(t-2\ell)$, the output of the receiver's matched filter is

$$\sum_\ell x_\ell q(t-2\ell)$$

where $q(t) = \psi(t) * \psi^*(-t)$. In particular, the output sampled at $2k+\delta$ gives

$$y_k = \sum_\ell x_\ell q(\delta + 2(k-\ell))$$

which is of the form $(q*x)_k$ with $q_k = q(\delta+2k)$. Since the Fourier transform of $q(t)$ is $|\Psi(f)|^2$, by the hint we see that $q(t) = 2\,\mathrm{sinc}(t)/(1-t^2)$, in particular,

$$q_k = 2\frac{\mathrm{sinc}(2k+\delta)}{1-(2k+\delta)^2} = \frac{2\sin(\pi\delta)}{\pi(2k+\delta)[1-(2k+\delta)^2]}.$$

(d) For $k=0$, we see $q_0 = 2\,\mathrm{sinc}(\delta)/(1-\delta^2) \approx 2$. For $k \neq 0$, as $|\delta| \ll 1$, we can bound, $|2k+\delta| \geq |k|$ and $(2k+\delta)^2 - 1 > k^2$. Thus,

$$|q_k| \leq \frac{2}{\pi}|\sin(\pi\delta)|\frac{1}{|k|^3},$$

so $\alpha = 2/\pi$, $\beta = 3$ is a possible choice. For i.i.d. $\{x_k\}$, the energy of the intersymbol-interference term $\sum_{\ell\neq 0} q_\ell x_{k-\ell}$ is $\sum_{\ell\neq 0} |q_\ell|^2 E[|x_{k-\ell}|^2] = \mathcal{E}\sum_{\ell\neq 0}|q_\ell|^2$. Using the bound on the $|q_k|$ above, we can upper bound this by

$$\frac{8\mathcal{E}}{\pi^2}\sin^2(\pi\delta)\sum_{k=1}^{\infty} k^{-6}.$$

The last sum can be upper bounded by $1 + \int_1^\infty t^{-6}\,dt = 6/5$, yielding a bound of the form (const)$\mathcal{E}\sin^2(\pi\delta)$. The important thing to note from this computation is that the variance of the additive error introduced to the signal due to the timing error is proportional to the square of the timing error, and also proportional to the energy of the signal. To ensure that the timing errors do not become the limiting factor of our communication system by dominating the additive noise due to the channel (of energy σ^2), one should make sure that $\delta^2\mathcal{E} \ll \sigma^2$.