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School of Computer and Communication Sciences

Handout 20 Advanced Digital Communications
Homework 9 December 6, 2010

PROBLEM 1.

(a) For the channel of problem 1, homework 8, show that the canonical factorization is

$$Q(D) + \frac{q_0}{\text{SNR}_{\text{MFB}}} = \gamma_0(1 - r_2 D^{-1})(1 - r_2^* D).$$

What is γ_0 in terms of a and b? Please do not do this from scratch. You have done much of the work for this in problem 2, HW7.

(b) Find $A(D)$ and $W(D)$ for the MMSE DFE.

(c) Give an expression for $\gamma_{\text{MMSE-DFE}}$ MMSE-DFE. Compute its values for $a = 0, .5, 1$ for the $E_x = 1$ and $\sigma^2 = 0.1$ Sketch $\gamma_{\text{MMSE-DFE}}$ as in problem 1, HW8. Compare with your sketches from problem 1, HW8.

Hint.

$$\gamma_{\text{MMSE-DFE}} = 10 \log_{10} \frac{\text{SNR}_{\text{MFB}}}{\text{SNR}_{\text{MMSE-DFE}}}$$

PROBLEM 2.

Consider the following discrete time channel:

$$y_k = x_k + x_{k-1} + z_k$$

where $\{z_k\}$ is i.i.d. gaussian with variance σ^2 and is independent of $\{x_k\}$. Moreover suppose that $\{x_k\}$, are indepedently and uniformely chosen from $\{A, -A\}$.

(a) Find the optimal estimator $\hat{x}_k(y_k)$, an estimator that uses only the current channel output. Compute its error probability. Hint: Consider the cases $x_{k-1} = x_k$ and $x_{k-1} \neq x_k$ separately.

(b) Find the optimal estimator $\hat{x}_k^{DFE}(y_k, \hat{x}_{k-1})$ that uses the previously estimated symbol to detect the last sent symbol, then:

(i) Find $Pr(\hat{x}_k \neq x_k)$ conditioned on $\hat{x}_{k-1} = x_{k-1}$

(ii) Find $Pr(\hat{x}_k \neq x_k)$ conditioned on $\hat{x}_{k-1} \neq x_{k-1}$

(iii) Find $Pr(\hat{x}_k \neq x_k)$ and compare it with the probability of error of the previous estimator.

PROBLEM 3. Consider the scalar discrete-time inter symbol interference channel,

$$y_k = \sum_{n=0}^{\nu} p_n x_{k-n} + z_k, \quad k = 0, \ldots, N-1, \tag{1}$$

where $z_k \sim \mathbf{C}\mathcal{N}(0, \sigma_z^2)$ and is i.i.d., independent of $\{x_k\}$. Let us employ a cyclic prefix as done in OFDM, *i.e.,*

$$x_{-l} = x_{N-1-l}, \quad l = 0, \ldots, \nu.$$

As done in class given the cyclic prefix,

$$\mathbf{y} = \begin{bmatrix} y_{N-1} \\ \vdots \\ y_0 \end{bmatrix} = \underbrace{\begin{bmatrix} p_0 & \dots & \dots & p_\nu & 0 & \dots & 0 & 0 \\ 0 & p_0 & \dots & p_{\nu-1} & p_\nu & 0 & \dots & 0 \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & \dots & \dots & 0 & p_0 & \dots & \dots & p_\nu \\ p_\nu & 0 & \dots & 0 & 0 & p_0 & \dots & p_{\nu-1} \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ p_1 & \dots & p_\nu & 0 & \dots & 0 & 0 & p_0 \end{bmatrix}}_{\mathbf{P}} \underbrace{\begin{bmatrix} x_{N-1} \\ \vdots \\ x_0 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} z_{N-1} \\ \vdots \\ z_0 \end{bmatrix}}_{\mathbf{z}}. \tag{2}$$

In the derivation of OFDM we used the property that

$$\mathbf{P} = \mathbf{V}^* \mathbf{D} \mathbf{V}, \tag{3}$$

where

$$\mathbf{V}_{p,q} = \frac{1}{\sqrt{N}} \exp\left(-j\frac{2\pi}{N}(p-1)(q-1)\right)$$

and $\mathbf{D}$ is the diagonal matrix with

$$\mathbf{D}_{l,l} = d_l = \sum_{n=0}^{\nu} p_n e^{-j\frac{2\pi}{N}nl}.$$

Using this we obtained

$$\mathbf{Y} = \mathbf{V}\mathbf{y} = \mathbf{D}\mathbf{X} + \mathbf{Z},$$

where $\mathbf{X} = \mathbf{V}\mathbf{x}$, $\mathbf{Z} = \mathbf{V}\mathbf{z}$. This yields the parallel channel result

$$\mathbf{Y}_l = d_l \mathbf{X}_l + \mathbf{Z}_l. \tag{4}$$

If the carrier synchronization is not accurate, then (1) gets modified as

$$y(k) = \sum_{n=0}^{\nu} e^{j2\pi f_0 k} p_n x_{k-n} + z_k, \quad k = 0, \dots, N-1 \tag{5}$$

where f_0 is the carrier frequency offset. If we still use the cyclic prefix for transmission, then (2) gets modified as

$$\underbrace{\begin{bmatrix} y(N-1) \\ \cdot \\ \cdot \\ y(0) \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} p_0 e^{j2\pi f_0(N-1)} & \dots & p_\nu e^{j2\pi f_0(N-1)} & 0 & \dots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & & \\ 0 & \ddots & \ddots & \ddots & e^{j2\pi f_0 \nu} p_0 & \dots & e^{j2\pi f_0 \nu} p_\nu \\ \ddots & \ddots & \ddots & \ddots & \ddots & & \\ e^{j2\pi f_0 0} p_1 & \dots & e^{j2\pi f_0 0} p_\nu & 0 & \dots & 0 & e^{j2\pi f_0 0} p_0 \end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix} x_{N-1} \\ \vdots \\ x_0 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} z_{N-1} \\ \vdots \\ z_0 \end{bmatrix}}_{\mathbf{z}} \tag{6}$$

i.e.,

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{z}$$

Note that

$$\mathbf{H} = \mathbf{S}\mathbf{P},$$

where $\mathbf{S}$ is a diagonal matrix with $\mathbf{S}_{l,l} = e^{j2\pi f_0(N-l)}$ and $\mathbf{P}$ is defined as in (2).

(a) Show that for $\mathbf{Y} = \mathbf{V}\mathbf{y}$, $\mathbf{X} = \mathbf{V}\mathbf{x}$,

$$\mathbf{Y} = \mathbf{G}\mathbf{X} + \mathbf{Z} \tag{7}$$

and prove that

$$\mathbf{G} = \mathbf{V}\mathbf{S}\mathbf{V}^*\mathbf{D}.$$

(b) If $f_0 \neq 0$, we see from part (a) that $\mathbf{G}$ is no longer a diagonal matrix and therefore we do not obtain the parallel channel result of (4). We get inter-carrier interference (ICI), *i.e.,* we have

$$\mathbf{Y}_l = \mathbf{G}_{l,l}\mathbf{X}_l + \underbrace{\sum_{q \neq l} \mathbf{G}(l,q)\mathbf{X}_q + \mathbf{Z}_l}_{\text{ICI + noise}}, \quad l = 0, \ldots, N-1,$$

which shows that the other carriers interfere with $\mathbf{X}_l$. Compute the SINR (signal-to-interference plus noise ratio). Assume $\{\mathbf{X}_l\}$ are i.i.d, with $\mathbb{E}|\mathbf{X}_l|^2 = \mathcal{E}_x$. You can compute the SINR for the particular l and leave the expression in terms of $\{G(l,q)\}$.

(c) Find the filter $\mathbf{W}_l$, such that the MMSE criterion is fulfilled,

$$\min_{\mathbf{W}_l} \mathbb{E}|\mathbf{W}_l^*\mathbf{Y} - \mathbf{X}_l|^2.$$

You can again assume that $\{\mathbf{X}_l\}$ are i.i.d with $\mathbb{E}|\mathbf{X}_l|^2 = \mathcal{E}_x$ and that the receiver knows $\mathbf{G}$. You can now state the answer in terms of $\mathbf{G}$.

(d) Find an expression for $\mathbf{G}_{l,q}$ in terms of $f_0, N, \{d_l\}$. Using Taylor series show that $\mathbf{G}_{l,q} \approx \theta c_{l,q}$ for small θ. What can you say about the ICI if $\theta << 1/N^2$?

Hint: Use the summation of the geometric series

$$\sum_{n=0}^{N-1} r^n = \frac{1 - r^N}{1 - r}.$$