ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout X Advanced Digital Communications
Homework 9 December 14, 2010

Problem 1. (a) From Problem 1 of HW8 we have:

$$
\begin{aligned}
q_0 &= 1 + aa^* && (1)\\
b &= q_0\left(1 + \frac{1}{SNR_{MFB}}\right) && (2)\\
r_2 &= \frac{-b + \sqrt{b^2 - 4aa^*}}{2a} && (3)\\
W_{MMSE-LE}(D) &= \frac{1}{a^* D^{-1} + b + aD} = \frac{1}{a}\frac{D}{(D - r_1)(D - r_2)} && (4)\\
Q(D) + \frac{q_0}{SNR_{MFB}} &= a^* D^{-1} + aD + q_0 + \frac{q_0}{SNR_{MFB}} && (5)\\
&= a^* D^{-1} + b + aD = \frac{1}{W_{MMSE-LE}(D)} && (6)\\
&= a\frac{(D - r_1)(D - r_2)}{D} && (7)\\
&= a(D - r_1)(1 - r_2 D^{-1}) && (8)\\
&= ar_1(Dr_2^* - 1)(1 - r_2 D^{-1}) && (9)\\
&= -ar_1(1 - r_2 D^{-1})(1 - r_2^* D) \quad (since \quad r_1 r_2^* = 1) && (10)
\end{aligned}
$$

$$
\begin{aligned}
\Longrightarrow \gamma_0 &= -ar_1 && (11)\\
&= a\frac{b + \sqrt{b^2 - 4aa^*}}{2a} && (12)\\
&= \frac{b + \sqrt{b^2 - 4aa^*}}{2} && (13)
\end{aligned}
$$

(b)

$$
\begin{aligned}
A(D) &= G(D) = 1 - r_2^* D && (14)\\
&= 1 - \frac{-b + \sqrt{b^2 - 4aa^*}}{2a^*}D && (15)\\
W(D) &= A(D)W_{MMSE-LE}(D) && (16)\\
&= \frac{A(D)}{\left(Q(D) + \frac{q_0}{SNR_{MFB}}\right)} && (17)\\
&= -\frac{1}{ar_1(1 - r_2 D^{-1})} && (18)
\end{aligned}
$$

(c)

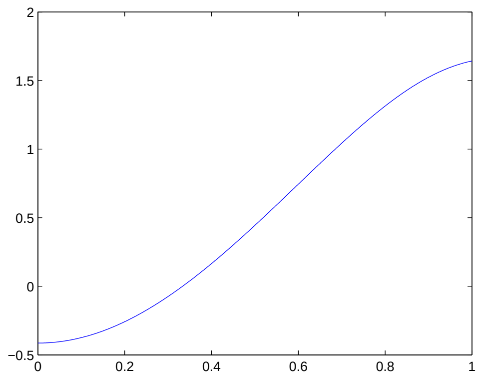
$$\begin{aligned}
SNR_{MMSE-DFE} &= \gamma_0 SNR_{MFB}/q_0 && (19)\\
\Longrightarrow \gamma_{MMSE-DFE} &= 10log_{10}\frac{q_0}{\gamma_0} && (20)\\
&= 10log_{10}\Big(\frac{2(1+aa^*)}{b+\sqrt{b^2-4aa^*}}\Big) && (21)\\
where,\quad b &= (1+aa^*)\big(1+\frac{1}{SNR_{MFB}}\big) && (22)\\
&= (1+aa^*)\big(1+\frac{N_0}{E_x(1+aa^*)}\big) && (23)\\
&= (1+aa^*)+0.1 = 1.1+aa^* && (24)
\end{aligned}$$

For $a = 0$, $\gamma = 10log_{10}(\frac{1}{1.1}) = -10log_{10}(1.1) = -0.41$

For $a = 0.5$, $\gamma = 10log_{10}(1.11) = 0.44$

For $a = 1$, $\gamma = 10log_{10}(1.46) = 1.64$

The plot for γ_{DFE} :



Problem 2.

(a) Since everything is symmetric we estimate 1 if $y_k > 0$ and 0 if $y_k \leqslant 0$. The error probability of such estimator is:

$$P_e = \Pr\{x_k = x_{k-1}\} \cdot \Pr\{z_k > 2A\} + \Pr\{x_k \neq x_{k-1}\} \cdot \Pr\{z_k > 0\} = \frac{1}{2}Q\left(\frac{2A}{\sigma}\right) + \frac{1}{4}$$

(b) The best decision rule is to estimate A if $y_k - \hat{x}_{k-1} > 0$ and $-A$ if $y_k - \hat{x}_{k-1} \leqslant 0$. The error probability of the estimator is now:

(i)

$$\Pr(\hat{x}_k \neq x_k | \hat{x}_{k-1} = x_{k-1}) = Q\left(\frac{A}{\sigma}\right)$$

(ii)

$$\begin{aligned}\Pr(\hat{x}_k \neq x_k|\hat{x}_{k-1} \neq x_{k-1}) &= \Pr(\hat{x}_k \neq x_k|\hat{x}_{k-1} \neq x_{k-1}, x_k \neq x_{k-1})\Pr(x_k \neq x_{k-1}) + \\ &\quad \Pr(\hat{x}_k \neq x_k|\hat{x}_{k-1} \neq x_{k-1}, x_k = x_{k-1})\Pr(x_k = x_{k-1}) \\ &= \frac{1}{2}Q\left(\frac{-A}{\sigma}\right) + \frac{1}{2}\cdot Q\left(\frac{3A}{\sigma}\right)\end{aligned}$$

(iii)

$$\begin{aligned}\Pr(\hat{x}_k \neq x_k) &= \Pr(\hat{x}_{k-1} \neq x_{k-1})\cdot\Pr(\hat{x}_k \neq x_k|\hat{x}_{k-1} \neq x_{k-1}) + \\ &\quad \Pr(\hat{x}_{k-1} = x_{k-1})\cdot\Pr(\hat{x}_k \neq x_k|\hat{x}_{k-1} = x_{k-1}) \\ &= \Pr(\hat{x}_k \neq x_k)\cdot\Pr(\hat{x}_k \neq x_k|\hat{x}_{k-1} \neq x_{k-1}) + \\ &\quad (1-\Pr(\hat{x}_k \neq x_k))\cdot\Pr(\hat{x}_k \neq x_k|\hat{x}_{k-1} = x_{k-1})\end{aligned}$$

By solving the equation we obtain:

$$\Pr(\hat{x}_k \neq x_k) = \frac{2Q\left(\frac{A}{\sigma}\right)}{1+3Q(\frac{A}{\sigma}) - Q\left(\frac{3A}{\sigma}\right)}$$

We can observe that as A increases the probability of error goes to 0 while without feedback the error probability is lower bounded by $\frac{1}{4}$.

Problem 3.

(a)

$$\mathbf{y} = \mathbf{Hx} + \mathbf{z}$$
$$\mathbf{Vy} = \mathbf{VSPx} + \mathbf{Vz}$$
$$\Rightarrow \quad \mathbf{Y} = \mathbf{VSV^*DVx} + \mathbf{Vz}$$
$$\Rightarrow \quad \mathbf{Y} = \underbrace{\mathbf{VSV^*D}}_{\mathbf{G}}\mathbf{X} + \mathbf{Z}$$
$$\Rightarrow \quad \mathbf{Y} = \mathbf{GX} + \mathbf{Z}$$

(b)

$$\mathbf{Y}_l = \mathbf{G}_{l,l}\mathbf{X}_l + \underbrace{\sum_{q\neq l}\mathbf{G}(l,q)\mathbf{X}_q + \mathbf{Z}_l}_{\text{ICI + noise}}, \quad l = 0,\dots,N-1,$$

Hence,

$$\text{SINR} = \frac{\mathbb{E}\left(|\mathbf{G}_{l,l}\mathbf{X}_l|^2\right)}{\mathbb{E}\left(\left|\sum_{q\neq l}\mathbf{G}_{l,q}\mathbf{X}_q\right|^2\right) + \mathbb{E}\,|Z_l|^2} = \frac{\mathcal{E}_x|\mathbf{G}_{l,l}|^2}{\mathcal{E}_x\sum_{q\neq l}|\mathbf{G}_{l,q}|^2 + \sigma_z^2}$$

(c)

$$\begin{aligned}\mathbb{E}(\mathbf{YY}^*) &= \mathbb{E}((\mathbf{GX}+\mathbf{Z})(\mathbf{X}^*\mathbf{G}^* + \mathbf{Z}^*)) \\ &= \mathcal{E}_x\mathbf{GG}^* + \mathbf{I}\sigma_z^2.\end{aligned} \tag{25}$$

$$\begin{aligned}\mathbb{E}(\mathbf{X}_l\mathbf{Y}^*) &= \mathbb{E}(\mathbf{X}_l(\mathbf{X}^*\mathbf{G}^* + \mathbf{Z}^*)) \\ &= \mathbb{1}_l^T\mathcal{E}_x\mathbf{G}^*,\end{aligned} \tag{26}$$

where $\mathbb{1}_l^T = \begin{bmatrix} 0 & \cdots & \underbrace{1}_{l^{\text{th}}\text{position}} & 0 & \cdots & 0 \end{bmatrix}$. Orthogonality principle implies,

$$\mathbb{E}((\mathbf{W}_l^*\mathbf{Y} - \mathbf{X}_l)\mathbf{Y}^*) = 0$$

$$\Rightarrow \mathbb{E}(\mathbf{W}_l^*\mathbf{Y}\mathbf{Y}^*) = \mathbb{E}(\mathbf{X}_l\mathbf{Y}^*)$$

$$\Rightarrow \mathbf{W}_l^* = \mathbb{E}(\mathbf{X}_l\mathbf{Y}^*)(\mathbb{E}(\mathbf{Y}\mathbf{Y}^*))^{-1}$$

Using equations 25,26 we get that,

$$\Rightarrow \mathbf{W}_l^* = \mathcal{E}_x \mathbb{1}_l^T \mathbf{G}^* (\mathcal{E}_x \mathbf{G}\mathbf{G}^* + \mathbf{I}\sigma_z^2)^{-1}$$

(d)

$$\mathbf{G}_{l,q} = (\mathbf{VS})_l (\mathbf{V}^*\mathbf{D})_q$$

where $(\mathbf{VS})_l$ denotes the l^{th} row of $\mathbf{VS}$ and $(\mathbf{V}^*\mathbf{D})_q$ denotes the q^{th} column of $\mathbf{V}^*\mathbf{D}$.

$$\mathbf{G}_{l,q} = \frac{1}{N} \begin{bmatrix} e^{j2\pi f_0(N-1)} & e^{j2\pi f_0(N-2)} e^{-j\frac{2\pi}{N}(l-1)} \cdots & e^{j2\pi f_0(N-N)} e^{-j\frac{2\pi}{N}(l-1)(N-1)} \end{bmatrix} \begin{bmatrix} d_q \\ d_q e^{j\frac{2\pi}{N}(q-1)} \\ \vdots \\ d_q e^{j\frac{2\pi}{N}(q-1)(N-1)} \end{bmatrix}$$

$$\Rightarrow \quad \mathbf{G}_{l,q} = \frac{d_q}{N} e^{j2\pi f_0(N-1)} \sum_{p=1}^{N} e^{\left(j\frac{2\pi}{N}(q-l) - j2\pi f_0\right)(p-1)}$$

By using the summation formula for the geometric series we get,

$$\mathbf{G}_{l,q} = \frac{d_q}{N} e^{j2\pi f_0(N-1)} \left[\frac{1 - e^{-j2\pi f_0 N}}{1 - e^{j\frac{2\pi}{N}(q-l-f_0 N)}} \right] \quad \text{for} \quad f_0 \neq 0.$$

The ICI is significant when $\mathbf{G}_{l,q}$ is comparable to $\mathbf{G}_{l,l}$. When $f_0 N$ is large then this could occur, i.e., there is significant time variation over the block.