ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 19 Advanced Digital Communications

Solutions - Homework 8 December 8, 2010

PROBLEM 1.

(a) For the Zero-forcing equalizer we have,

$$W_{ZFE}(D) = \frac{1}{Q(D)} = \frac{1}{a^*D^{-1} + 1 + aa^* + aD} \tag{1}$$

For the MMSE-LE we have calculated in Problem 1- Part (c) that,

$$\begin{aligned} W_{MMSE-LE}(D) &= \frac{1}{Q(D) + \frac{N_0}{E_x}} &(2)\\ &= \frac{1}{a^*D^{-1} + aD + 1 + aa^* + \frac{N_0}{E_x}} &(3)\\ &= \frac{1}{a^*D^{-1} + b + aD} &(4)\end{aligned}$$

Since

$$b = q_0(1 + \frac{1}{SNR_{MFB}}) = q_0 + \frac{N_0}{E_x}$$

(b) Notice the pole-like behaviour for a=0.9, and difference in power between W_{ZFE} and $W_{MMSE-LE}$

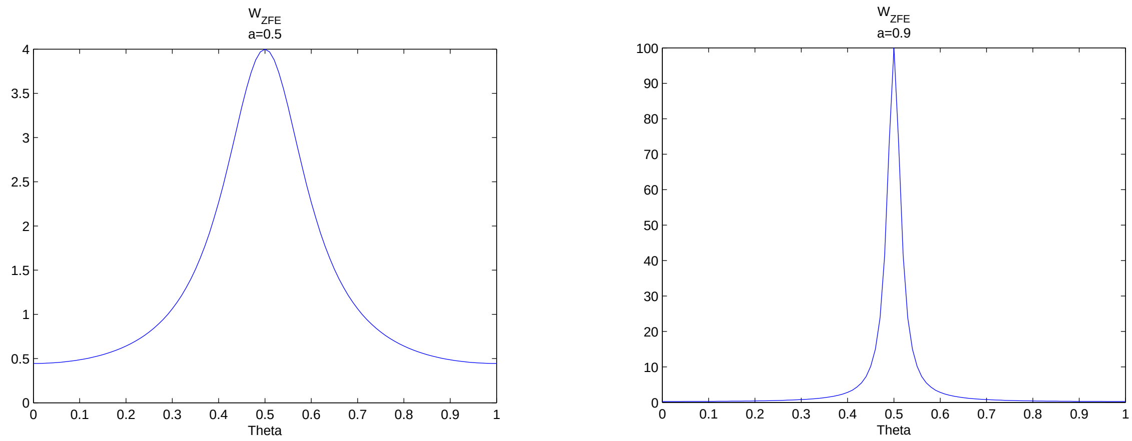

Figure 1: W_{ZFE} for $a = 0.5$ (left) and $a = 0.9$ (right)

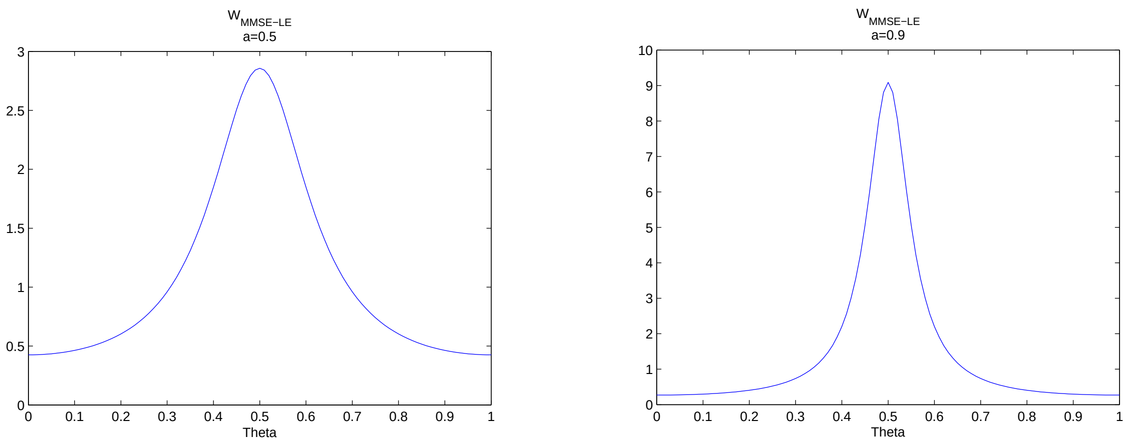

Figure 2: $W_{MMSE-LE}$ for $a = 0.5$ (left) and $a = 0.9$ (right)

(c) The roots are given by:

$$r_1 = \frac{-b - \sqrt{b^2 - 4aa^*}}{2a} \tag{5}$$

$$r_2 = \frac{-b + \sqrt{b^2 - 4aa^*}}{2a} \tag{6}$$

$b^2 - 4aa^*$ is always real since b^2 and aa^* is real.

$$b^2 - 4aa^* = [(1 + aa^*)(1 + \frac{1}{SNR_{MFB}})]^2 - 4aa^* \tag{7}$$

$$\geq (1 + aa^*)^2 - 4aa^* = (1 - aa^*)^2 > 0 \tag{8}$$

(d) $W_{MMSE-LE}(D)$ can be written as:

$$W_{MMSE-LE}(D) = \frac{1}{a^* D^{-1} + b + aD}$$

By regrouping and expanding it into partial fractions we obtain:

$$W_{MMSE-LE}(D) = \frac{1}{a} \frac{D}{(D - r_1)(D - r_2)} \tag{9}$$

$$= \frac{1}{a}[\frac{A}{D - r_1} + \frac{B}{D - r_2}] \tag{10}$$

$$= \frac{1}{a(r_1 - r_2)}(\frac{r_1}{D - r_1} - \frac{r_2}{D - r_2}) \tag{11}$$

(e) We know that $|r_2| < |r_1|$ and $r_1 r_2^* = 1$. So, $|r_2| < 1$. We can rewrite the expansion in part (d):

$$W(D) = \frac{1}{a(r_1 - r_2)}\left[\frac{-1}{1 - D/r_1} - \frac{r_2}{D}(\frac{1}{1 - \frac{r_2}{D}})\right]$$

$$= \frac{1}{a(r_1 - r_2)}\left[-(1 + (\frac{D}{r_1}) + (\frac{D}{r_1})^2 + \ldots) - \frac{r_2}{D}(\frac{1}{1 - \frac{r_2}{D}})\right]$$

Thus, for the MMSE-LE we have:

$$w_0 = \frac{-1}{a(r_1 - r_2)} = \frac{1}{\sqrt{b^2 - 4aa^*}}$$

For the ZFE, we take $\frac{1}{SNR_{MFB}} = 0$ and thus $b = 1 + aa^*$ to obtain

$$w_0 = \frac{1}{1 - aa^*}$$

(f)

$$SNR_{MFB} = \frac{q_0 E_x}{N_0} = 10(1 + aa^*)$$

$$\begin{aligned} SNR_{ZFE} &= \frac{E_x}{\sigma^2_{ZFE}} = \frac{E_x}{\sigma^2 w_0} && (12) \\ &= 10(1 - aa^*) && (13) \\ {\sigma_{ZFE}}^2 &= \sigma^2 w_0 = \frac{\sigma^2}{(1 - aa^*)} && (14) \\ &= \frac{1}{10(1 - aa^*)} && (15) \\ \gamma_{ZFE} &= 10 \log \frac{(1 + aa^*)}{(1 - aa^*)} && (16) \end{aligned}$$

Similarly,

$$\begin{aligned} SNR_{MMSE-LE} &= \frac{E_x}{{\sigma_{MMSE-LE}}^2} = \frac{E_x}{\sigma^2 w_0} && (17) \\ &= 10\sqrt{b^2 - 4aa^*} = 10\sqrt{(b^2 - 4aa^*)} && (18) \\ \sigma^2_{MMSE-LE} &= \sigma^2 w_0 = \frac{\sigma^2}{\sqrt{b^2 - 4aa^*}} && (19) \\ &= \frac{1}{10\sqrt{b^2 - 4aa^*}} && (20) \\ \gamma_{MMSE-LE} &= 10 \log \frac{1 + aa^*}{\sqrt{b^2 - 4aa^*}} && (21) \\ &= 10 \log \frac{1 + aa^*}{\sqrt{(1 + aa^* + 0.1)^2 - 4aa^*}} && (22) \end{aligned}$$

(g) Using what we have found in part (f)

$$\gamma_{ZFE} = \begin{cases} 10 \log 1 = 0 & , \quad a = 0 \\ 10 \log \frac{1.25}{0.75} = 2.218 & , \quad a = 0.5 \\ 10 \log \frac{2}{0} = \infty & , \quad a = 1 \end{cases}$$

$$\gamma_{MMSE-LE} = \begin{cases} 10 \log \frac{1}{\sqrt{1.21}} = -0.41 & , \quad a = 0 \\ 10 \log \frac{1.25}{0.907} = 1.39 & , \quad a = 0.5 \\ 10 \log \frac{2}{0.64} = 4.95 & , \quad a = 1 \end{cases}$$

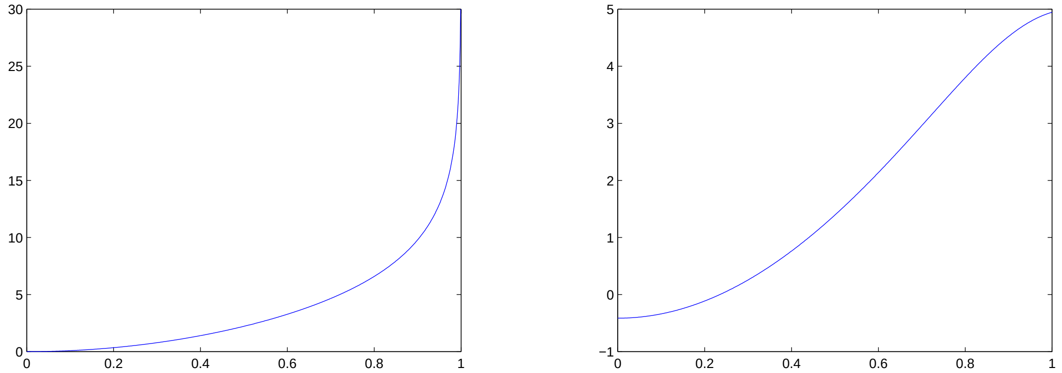

Figure 3: γ_{ZFE} (left) and $\gamma_{MMSE-LE}$ (right)