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School of Computer and Communication Sciences

Handout 18 Advanced Digital Communications
Homework 7 (Solutions) December 1, 2010

PROBLEM 1.

(a) Via orthogonality, we can write

$$E[(U(D) - \hat{U}(D))Y^*(D^{-*})] = 0$$

where, $U(D) = H(D)X(D)$ and $\hat{U}(D) = W(D)Y(D)$

Thus,

$$E[(H(D)X(D) - W(D)Y(D))Y^*(D^{-*})] = 0 \quad (1)$$
$$E[(H(D)X(D)Y^*(D^{-*})] = E[W(D)Y(D))Y^*(D^{-*})] \quad (2)$$

Using $Y^*(D^{-*}) = Q^*(D^{-*})X^*(D^{-*}) + Z^*(D^{-*})$, for the R.H.S. we get :

$$\begin{aligned} E[(H(D)X(D)Y^*(D^{-*})] &= H(D)Q^*(D^{-*})E_x + \underbrace{E[H(D)X(D)Z^*(D^{-*})]}_{0} \\ &= H(D)Q(D)E_x \quad (3) \end{aligned}$$

Since, X_k and Z_k are independent and $Q(D) = Q^*(D^{-*})$

and for the L.H.S. :

$$E[W(D)Y(D))Y^*(D^{-*})] = W(D)[Q^*(D^{-*})E[Y(D)X^*(D^{-*})] + E[Y(D)Z^*(D^{-*}))]]$$
$$= W(D)(Q(D)(E[Q(D)X(D)] + \underbrace{E[Z(D)X^*(D^{-*})]}_{0})] + E[(Q(D)X(D) + Z(D))Z^*(D^{-*})])$$
$$= W(D)(Q^2(D)E_x + E[(Z(D)Z^*(D^{-*})] + \underbrace{E[Q(D)X(D)Z^*(D^{-*})]}_{0}$$
$$= W(D)(Q^2(D)E_x + N_0Q(D))$$

Thus,

$$W(D)(Q^2(D)E_x + N_0Q(D)) = H(D)Q(D)E_x \quad (4)$$
$$W(D) = \frac{H(D)E_x}{Q(D)E_x + N_0} \quad (5)$$

(b)

$$\begin{aligned} & S_E(D) = E[E(D)E^*(D^{-*})] \quad (6) \\ = \;& E[(H(D)X(D) - W(D)Y(D))(H^*(D^{-*})X^*(D^{-*}) - W^*(D^{-*})Y^*(D^{-*}))] \quad (7) \\ = \;& H(D)H^*(D^{-*})E_x + W(D)W^*(D^{-*})(Q^2(D)E_x + N_0Q(D)) \\ & -W(D)Q(D)H^*(D^{-*})E_x - W^*(D^{-*})Q(D)H(D)E_x \quad (8) \end{aligned}$$

Substituting the W(D) found in part a, we obtain

$$S_E(D) = \frac{H(D)H^*(D^{-*})N_0E_x}{Q(D)E_x + N_0} \quad (9)$$

(c) If H(D)=1, the operation performed becomes a MMSE linear estimation with

$$W(D) = \frac{E_x}{Q(D)E_x + N_0}$$

PROBLEM 2. (a) We want to minimized the MSE,

$$\text{MSE} = E(x^2) + E(\hat{x}^2) - 2E(x\hat{x})$$

so,

$$\text{MSE} = E(x^2) + a^2|c^t h|^2 E(x^2) + a^2\frac{N_0}{2} - 2a|c^t h|E(x^2)$$

By differentiating with respect to a and setting it equal to zero we will have the result.

(b) The value of MSE is

$$\frac{\frac{N_0}{2}E(x^2)}{|c^t h|^2 E(x^2) + \frac{N_0}{2}}$$

By simplifying above we will have the result.

(c) To minimize the MSE we should maximize the denominator, by Cauchy-Schwarz inequality we know

$$|c^t h|^2 \leq ||h||^2$$

and it satisfies the equality if h and c are proportional to each other, putting the norm constraint we will have $c = \frac{h}{|h|}$.

PROBLEM 3. We set $\hat{x} = A^t Y$. By orthogonality principle we have,

$$E((X - \hat{X})Y^*) = 0$$

So we have

$$E(XY^*) = E(A^t Y Y^*)$$

So in general we will have

$$\begin{bmatrix} E(Y_1 Y_1^*) & E(Y_1 Y_2^*) \\ E(Y_2 Y_1^*) & E(Y_2 Y_2^*) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} E(XY_1^*) \\ E(XY_2^*) \end{bmatrix}$$

Which is

$$\begin{bmatrix} \mathcal{E}_x + E(Z_1 Z_1^*) & \mathcal{E}_x + E(Z_1 Z_2^*) \\ \mathcal{E}_x + E(Z_2 Z_1^*) & \mathcal{E}_x + E(Z_2 Z_2^*) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \mathcal{E}_x \\ \mathcal{E}_x \end{bmatrix}$$

We solve the above linear equations for each case

(a)

$$a_1 = a_2 = \frac{\mathcal{E}_x}{2\mathcal{E}_x + 1}$$

(b)

$$a_1 = a_2 = \frac{\mathcal{E}_x}{2\mathcal{E}_x + 1 + \frac{\sqrt{2}}{2}}$$

(c) In this case the set of linear equations has infinite solutions, any of them is good for us. In particular

$$a_1 = a_2 = \frac{\mathcal{E}_x}{2(\mathcal{E}_x + 1)}$$

Problem 4. (a)

$$\widehat{X}_a = \frac{H_a^*\sigma_x^2}{H_aH_a^*\sigma_x^2+\sigma_a^2}Y_a, \quad \widehat{X}_b = \frac{H_b^*\sigma_x^2}{H_bH_b^*\sigma_x^2+\sigma_b^2}Y_b.$$

$$\begin{aligned}
P_a &= \sigma_x^2 - H_a^*\sigma_x^2(H_aH_a^*\sigma_x^2+\sigma_a^2)^{-1}H_a\sigma_x^2, \\
&= \frac{\sigma_x^2\sigma_a^2}{|H_a|^2\sigma_x^2+\sigma_a^2}, \\
P_b &= \sigma_x^2 - H_b^*\sigma_x^2(H_bH_b^*\sigma_x^2+\sigma_b^2)^{-1}H_b\sigma_x^2, \\
&= \frac{\sigma_x^2\sigma_b^2}{|H_b|^2\sigma_x^2+\sigma_b^2}.
\end{aligned}$$

(b) Using the identities

$$\begin{aligned}
\widehat{X}_a &= (\frac{1}{\sigma_x^2}+\frac{H_aH_a^*}{\sigma_a^2})^{-1}\frac{H_a^*}{\sigma_a^2}Y_a, \\
\Rightarrow (\frac{1}{\sigma_x^2}+\frac{H_aH_a^*}{\sigma_a^2})\widehat{X}_a &= \frac{H_a^*}{\sigma_a^2}Y_a, \\
\Rightarrow P_a^{-1}\widehat{X}_a &= \frac{H_a^*}{\sigma_a^2}Y_a.
\end{aligned}$$

Similarly,

$$P_b^{-1}\widehat{X}_b = \frac{H_b^*}{\sigma_b^2}Y_b.$$

(c) Now

$$\widehat{X} = \begin{bmatrix} H_a^* & H_b^* \end{bmatrix}\sigma_x^2 \begin{bmatrix} H_aH_a^*\sigma_x^2+\sigma_a^2 & H_aH_b^*\sigma_x^2 \\ H_bH_a^*\sigma_x^2 & H_bH_b^*\sigma_x^2+\sigma_b^2 \end{bmatrix}^{-1}\begin{bmatrix} Y_a \\ Y_b \end{bmatrix}$$

$$P = \mathcal{E}_x - \sigma_x^2\begin{bmatrix} H_a^* & H_b^* \end{bmatrix}\begin{bmatrix} H_aH_a^*\sigma_x^2+\sigma_a^2 & H_aH_b^*\sigma_x^2 \\ H_bH_a^*\sigma_x^2 & H_bH_b^*\sigma_x^2+\sigma_b^2 \end{bmatrix}^{-1}\begin{bmatrix} H_a \\ H_b \end{bmatrix}\sigma_x^2.$$

Using the matrix identities

$$\mathbf{H} = \begin{bmatrix} H_a \\ H_b \end{bmatrix}, \mathbf{R_v} = \begin{bmatrix} \sigma_a^2 & 0 \\ 0 & \sigma_b^2 \end{bmatrix}, \mathbf{R_x} = \sigma_x^2.$$

We get

$$\begin{aligned}
\widehat{X} &= (\frac{1}{\sigma_x^2}+\begin{bmatrix} H_a^* & H_b^* \end{bmatrix}\begin{bmatrix} \sigma_a^2 & 0 \\ 0 & \sigma_b^2 \end{bmatrix}^{-1}\begin{bmatrix} H_a \\ H_b \end{bmatrix})^{-1}\begin{bmatrix} H_a^* & H_b^* \end{bmatrix}\begin{bmatrix} \sigma_a^2 & 0 \\ 0 & \sigma_b^2 \end{bmatrix}^{-1}\begin{bmatrix} Y_a \\ Y_b \end{bmatrix}, \\
\Rightarrow &\quad P^{-1}\widehat{X} = \frac{H_a^*}{\sigma_a^2}Y_a + \frac{H_b^*}{\sigma_b^2}Y_b = P_a^{-1}\widehat{X}_a + P_b^{-1}\widehat{X}_b.
\end{aligned}$$

Now

$$\begin{aligned}
P^{-1} &= (\frac{1}{\sigma_x^2}+\begin{bmatrix} H_a^* & H_b^* \end{bmatrix}\begin{bmatrix} \sigma_a^2 & 0 \\ 0 & \sigma_b^2 \end{bmatrix}^{-1}\begin{bmatrix} H_a \\ H_b \end{bmatrix}), \\
&= (\frac{1}{\sigma_x^2}+\frac{H_a^*H_a}{\sigma_a^2}+\frac{H_b^*H_b}{\sigma_b^2}), \\
&= P_a^{-1}+P_b^{-1}-\frac{1}{\sigma_x^2}.
\end{aligned}$$