ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 13 — Advanced Digital Communications

Homework 6 — November 18, 2010

Problem 1.

In this exercise the receiver is not performing matched filtering. It is easy to see that the shifts of $g_{\text{TX}}(t)$ are not suitable for such kind of approach.

a) First compute the fourier transform of $g_{\text{TX}}(t)$:

$$\begin{aligned}
G_{\text{TX}}(f) &= T \cdot \operatorname{rect}(Tf) * (2T \operatorname{rect}(2Tf)) \\
&= 2T^2 \cdot \left(\operatorname{rect}(2Tf) * \delta\left(f - \frac{1}{4T}\right) + \operatorname{rect}(2Tf) * \delta\left(f + \frac{1}{4T}\right)\right) * \operatorname{rect}(2Tf) \\
&= 2T^2 \cdot \left(\frac{1}{2T}\Lambda(2T \cdot f) * \delta\left(f - \frac{1}{4T}\right) + \frac{1}{2T}\Lambda(2Tf) * \delta\left(f + \frac{1}{4T}\right) \right) \\
&= T \cdot \Lambda\left(2T \cdot f - \frac{1}{4T}\right) + T \cdot \Lambda\left(2Tf + \frac{1}{4T}\right)
\end{aligned}$$

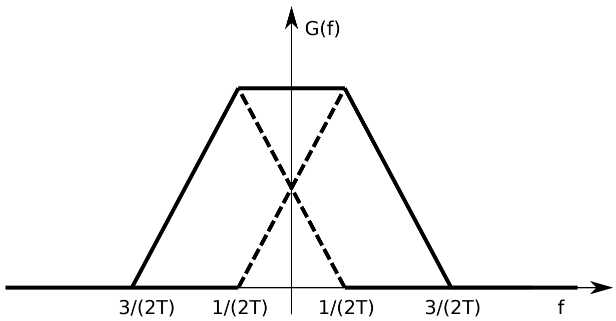

Figure 1: The Fourier transform of $g_{\text{TX}}(f)$

Then we check if Nyquist condition (the first we studied in the lecture) applies for symbol rate $1/T$:

$$h(f) = \sum_{k=-\infty}^{\infty} G_{\text{TX}}\left(f + \frac{k}{T}\right) = T$$

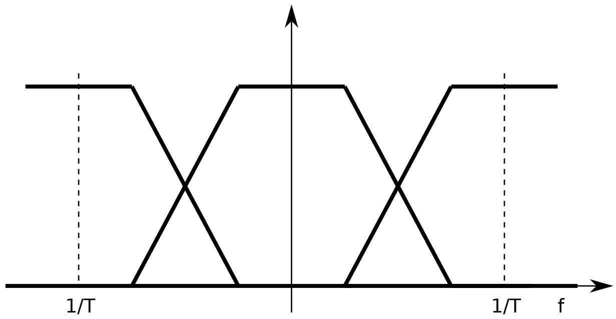

Figure 2: Superposition of shifted versions of $G_{\text{TX}}(f)$

Alternatively we can observe that the function is one at zero and is zero at every integer multiple of the period.

b) Since the bitrate is 6 Mbit/s and each symbol carried $\log_2(8) = 3$ bits we have a symbol rate of $1/T = 2 \cdot 10^6$ symbols/s (Baud), therefore the minimum bandwidth is 3 MHz.

c) If we sample the input signal at $t = 0.5 + kT$ (without performing matched filtering) we easily see that $y_n = x_n - \frac{1}{2}x_{n-1} - \frac{1}{4}x_{n-2}$. This means that the channel has memory $l = 2$ and therefore the number of states is $8^2 = 64$.

Problem 2.

The trellis for MLSE estimation is the following:

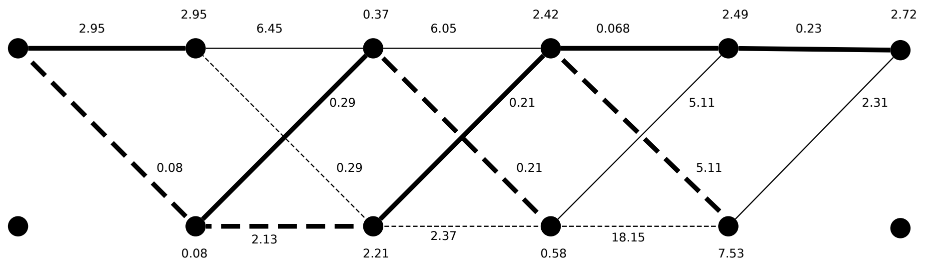

Figure 3: Trellis for MLSE estimate

where we used $|y_n - (s_{n+1} + s_n)|^2$ as edge metric. The decoded sequence is therefore $\{-1, -1, +1, +1, +1\}$.

When computing BCJR we have the following values for $\gamma_n(i,j)$ (edges) and $\alpha_n(i)$ (vertices):

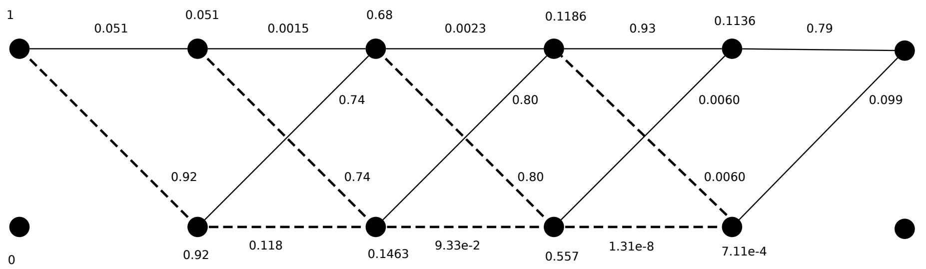

Figure 4: Computation of $a_n(i)$

and the following values for $\beta_n(i)$ (vertices):

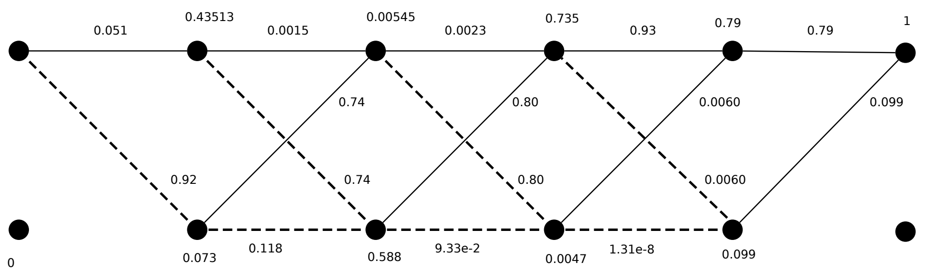

Figure 5: Computation of $\beta_n(i)$

where we use $\gamma_n(i,j) = \exp(-|y_n - (i+j)|^2)$.

By computing $\text{score}(x,n) = \beta_{n+1}(x) \cdot \gamma_n(-x,x) \cdot \alpha_n(-x) + \beta_{n+1}(x) \cdot \gamma_n(x,x) \cdot \alpha_n(x)$ we obtain the following result:

n	1	2	3	4	5
$+1$	0.023330	0.003895	0.090474	0.093155	
-1	0.069898	0.089333	0.002753	0.000073	
$\hat{x}(n)$	-1	-1	$+1$	$+1$	$+1$

Table 1: Score for each possible bit

We observe that the two algorithms give a different estimate for bits 2 and 3.

Problem 3.

i. 1.
$$E[\hat{x}|x] = E[\boldsymbol{a}^T\boldsymbol{h}x + \boldsymbol{a}^T z|x] = E[\boldsymbol{a}^T\boldsymbol{h}x|x] \Leftrightarrow \boldsymbol{a}^T\boldsymbol{h}x = x \Leftrightarrow \boldsymbol{a}^T\boldsymbol{h} = 1$$

2. First observe that:
$$\hat{x} = \boldsymbol{a}^T\boldsymbol{y} = x - \boldsymbol{a}^T\boldsymbol{z}$$
then:
$$x - \hat{x} = -\boldsymbol{a}^T\boldsymbol{z}$$
Therefore:
$$E[|x-\hat{x}|^2] = E[(\boldsymbol{a}^T\boldsymbol{z})^2] = \boldsymbol{a}^T E[\boldsymbol{z}\boldsymbol{z}^T]\boldsymbol{a} = \boldsymbol{a}^T I\boldsymbol{a} = \boldsymbol{a}^T\boldsymbol{a} = \sigma^2_{\text{unbiased}}$$
So we need to minimize $\boldsymbol{a}^T\boldsymbol{a}$ such that $\boldsymbol{a}^T\boldsymbol{h} = 1$. The solution to the minimization problem is $\boldsymbol{a} = (\boldsymbol{h}^T\boldsymbol{h})^{-1}\boldsymbol{h}^T$. In this case $\sigma^2_{\text{unbiased}}$=$\boldsymbol{a}^T\boldsymbol{a} = (\boldsymbol{h}^T\boldsymbol{h})^{-1}$.
We can indeed verify the solution by observing that:
$$(\boldsymbol{a}^T\boldsymbol{a})(\boldsymbol{h}^T\boldsymbol{h}) \geqslant |\boldsymbol{a}^T\boldsymbol{h}|^2 = 1 \Rightarrow \boldsymbol{a}^T\boldsymbol{a} \geqslant (\boldsymbol{h}^T\boldsymbol{h})^{-1}$$
with equality when $\boldsymbol{a}$ is chosen as proposed.

ii. First observe that:
$$\hat{x} = \boldsymbol{a}^T\boldsymbol{y} = cx - \boldsymbol{a}^T\boldsymbol{z}$$
then:
$$x - \hat{x} = (1-c)x - \boldsymbol{a}^T\boldsymbol{z}$$
Therefore,
$$E[|x-\hat{x}|^2] = E\left[\left[(1-c)x - (\boldsymbol{a}^T\boldsymbol{z})\right]^2\right] = (1-c)^2\mathcal{E} + \boldsymbol{a}^T\boldsymbol{a} = \sigma^2_{\text{min}}(c)$$
Since $(1-c)^2\mathcal{E}$ is fixed we just need to minimize $\boldsymbol{a}^T\boldsymbol{a}$ such that $\boldsymbol{a}^T\boldsymbol{h} = c$. The solution to the minimization problem is $\boldsymbol{a} = c(\boldsymbol{h}^T\boldsymbol{h})^{-1}\boldsymbol{h}^T$. In this case:
$$\sigma^2_{\text{min}}(c) = (\boldsymbol{h}^T\boldsymbol{h})^{-1}c^2 + (c-1)^2\mathcal{E}$$
We can find the c that minimizes $\sigma^2_{\text{min}}(c)$ by finding the zero of the derivative. We obtain:
$$c = \frac{\mathcal{E}}{(\boldsymbol{h}^T\boldsymbol{h})^{-1} + \mathcal{E}}$$
The minimal $\sigma^2_{\text{min}}(c)$ is therefore:
$$\sigma^2_{\text{min}} = \frac{\mathcal{E}}{\boldsymbol{h}^T\boldsymbol{h}\mathcal{E} + 1}$$

iii. In the first case we have:

$$\frac{\mathcal{E}}{\sigma^2_{\text{unbiased}}} = \mathcal{E}\boldsymbol{h}^T\boldsymbol{h}$$

In the second case:

$$\frac{\mathcal{E}}{\sigma^2_{\text{min}}} = \mathcal{E}\boldsymbol{h}^T\boldsymbol{h} + 1$$

iv. We notice that in both cases $\boldsymbol{a}^T = c \cdot (\boldsymbol{h}^T\boldsymbol{h})^{-1}\boldsymbol{h}^T$ with $c = 1$ for the first part. The probability of making an error is:

$$\begin{aligned}
\Pr\{\hat{x} = k | x = -k\} &= \Pr\{\hat{x} = 1 | x = -1\} = \Pr\{-\boldsymbol{a}^T\boldsymbol{h} + \boldsymbol{a}^T\boldsymbol{z} > 0\} \\
&= \Pr\{\boldsymbol{a}^T\boldsymbol{z} > \boldsymbol{a}^T\boldsymbol{h}\} = \Pr\{c \cdot (\boldsymbol{h}^T\boldsymbol{h})^{-1}\boldsymbol{h}^T\boldsymbol{z} > c \cdot (\boldsymbol{h}^T\boldsymbol{h})^{-1}\boldsymbol{h}^T\boldsymbol{h}\} = \\
&= \Pr\{(\boldsymbol{h}^T\boldsymbol{h})^{-1}\boldsymbol{h}^T\boldsymbol{z} > 1\}
\end{aligned}$$

which is independent of c. This means that both estimators have the same error probability.