ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 8 Advanced Digital Communications
Homework 5 October 25, 2010

PROBLEM 1. Let $z(t)$ be a stationary zero-mean Gaussian noise with power spectral density $S(f)$. Let

$$z_{bb}(t) = LPF\left[z(t)e^{-2j\pi f_c t}\right] \tag{1}$$

where LPF is a low pass filter that passes signals in the frequency range $[-W/2, W/2]$. The process $z_{bb}(t)$ is the baseband equivalent of $z(t)$.

Last week in class we used a heuristic argument that suggested that the stationarity of $z(t)$ together with the $W \ll f_c$ should imply that $z_{bb}(t)$ is a circularly symmetric Gaussian random process. This exercise gives a precise proof of this argument.

(a) Why is $z_{bb}(t)$ a complex Gaussian process? What is its mean?

(b) Recall that for complex Gaussian processes circular symmetry is equivalent to being zero mean and having pseudo-covariance J equal to zero. (For a complex random process W, $J_W(s,t) = E[W(s)W(t)]$ in analogy with the formula $J_{ik} = E[W_i W_k]$ for the vector case.). Show that

$$J(s,t) = \int S(f)H(f - f_c)H(-f - f_c)e^{-j2\pi f(t-s)}e^{-j2\pi f_c(t+s)}df,$$

where H is the frequency response of the low pass filter.

(c) Show that if $W < f_c$ then $J(s,t) = 0$. Conclude that z_{bb} is a circularly symmetric complex Gaussian process whenever $W < f_c$.

PROBLEM 2. Let us consider a real valued passband random process

$$X(t) = X_I(t)\cos(\omega_c t) - X_Q(t)\sin(\omega_c t), \tag{2}$$

at a carrier frequency of ω_c. The processes $\{X_I(t)\}$ and $\{X_Q(t)\}$ are independent stationary zero-mean Gaussian random processes with,

$$\begin{aligned} \mathbb{E}\left(X_I(t)X_I(t-\tau)\right) &= r_I(\tau) \\ \text{and} \quad \mathbb{E}\left(X_Q(t)X_Q(t-\tau)\right) &= r_Q(\tau). \end{aligned}$$

We assume that

$$\begin{aligned} r_I(\tau) = r_Q(\tau) &= e^{-2|\tau|} \\ \text{and} \quad \mathbb{E}\left(X_I(t)X_Q(t-\tau)\right) &= 0 \quad \forall \tau. \end{aligned}$$

1. Find $r_x(\tau) = \mathbb{E}\left(X(t)X(t-\tau)\right)$ in terms of $r_I(\tau) = r_Q(\tau)$ as a function of ω_c.

2. Find the correlation function for the baseband equivalent of $X(t)$ in equation (2), i.e., find
$$\mathbb{E}\left(X_{bb}(t)X_{bb}^*(t-\tau)\right) = r_{bb}(\tau).$$

3. We sample $X_{bb}(t)$ at times $t = k$, i.e., at integer times,

$$Y(k) = X_{bb}(k).$$

Hence

$$\mathbb{E}\left(Y(k)Y^*(k-l)\right) = \mathbb{E}\left(X_{bb}(k)X_{bb}^*(k-l)\right) = r_{bb}(l).$$

We want to find a base-band filter to whiten this $\{Y(k)\}$ process. Compute such a causal whitening filter.

Problem 3. Consider transmission over an ISI channel and suppose that the channel after matched filtering is

$$Y(D) = Q(D)X(D) + Z(D).$$

Suppose that $Q(D) = \sum_k q_k D^k$, where q_k is given by is given as the following

$$q_k = \begin{cases} 2^{-\frac{|k|-1}{2}} & \text{if } k \text{ is odd,} \\ \frac{5}{3}2^{-\frac{|k|}{2}} & \text{if } k \text{ is even.} \end{cases}$$

and the noise spectra is $S_Z(D) = N_0 Q(D)$. Find the whitening filter $W(D)$ to whiten the noise. Choose the filter so that the resulting communication channel after whitening is causal.

Problem 4. Consider transmission over an ISI channel with the transmitted signal being $x(t) = \sum_k x_k \varphi(t - kT)$ with

$$\varphi(t) = \frac{1}{\sqrt{T}} \operatorname{sinc}\left(\frac{t}{T}\right)$$

for symbol duration of T. Suppose that the ISI channel is characterized by its impulse response $h(t) = \delta(t) - 0.4\delta(t - T)$ and assume that the additive white Gaussian noise has power spectral density N_0.

(a) Determine the pulse response $\psi(t)$.

(b) Find $\| \psi \|$ and $\tilde{\varphi}(t) = \frac{\psi(t)}{\|\psi\|}$.

(c) Find the autocorrelation function of the noise after sampling the output of the matched filter. Find the whitening filter such that the resulting channel is causal.

(d) Suppose now that $x_i \in \{+1, -1\}$ (this is called binary antipodal signaling) and that $N_0 = 0.49$. Let the output of the whitened matched filter be $\{0.7, -1.1, -0.2, 0.9, -0.6, 0.9\}$. Find the maximum likelihood sequence using the Viterbi algorithm. Assume the initial and last states are $+1$.