ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 12 — Advanced Digital Communications

Homework 5 (Solution) — November 10, 2010

PROBLEM 1.

(a) $z(t)$ is a zero-mean Gaussian process. Down-converting to pass-band and applying a low-pass filter in the frequency range $[-W/2, W/2]$ does not change its statistical properties. So, $z_{bb}(t)$ is also a zero-mean Gaussian process.

(b)

$$z_{bb}(t) = \int h(t-\tau)z(\tau)e^{-j2\pi f_c \tau}d\tau. \tag{1}$$

$$\begin{aligned}
J(s,t) &= E[z_{bb}(s)z_{bb}(t)] = \int\int h(t-\tau)h(s-\delta)E[z(\tau)z(\delta)]e^{-j2\pi f_c(\tau+\delta)}d\tau d\delta & (2)\\
&= \int\int h(t-\tau)h(s-\delta)R(\delta-\tau)e^{-j2\pi f_c(\tau+\delta)}d\tau d\delta & (3)\\
&= \int\int\int h(t-\tau)h(s-\delta)S(f)e^{j2\pi f(\delta-\tau)}e^{-j2\pi f_c(\tau+\delta)}d\tau d\delta df & (4)\\
&= \int S(f)H(f-f_c)H(-f-f_c)e^{-j2\pi f(t-s)}e^{-j2\pi f_c(t+s)}df & (5)
\end{aligned}$$

(c) Observe that $H(f-f_c)$ and $H(-f-f_c)$ are shifted Fourier domain representations of the LPF with bandwidth W. So, if $f_c > W/2$, then the value of $J(s,t)$ is always 0.

PROBLEM 2.

(a)

$$\begin{aligned}
r_X(\tau) &= \mathbb{E}[X_I(t)X_I(t-\tau)]\cos\omega_c t\cos\omega_c(t-\tau) + \mathbb{E}[X_Q(t)X_Q(t-\tau)]\sin\omega_c t\sin\omega_c(t-\tau)\\
&= r_I(\tau)\left(\cos\omega_c t\cos\omega_c(t-\tau) + \sin\omega_c(t)\sin\omega_c(t-\tau)\right)\\
&= r_I(\tau)\left(\cos(\omega_c t - \omega_c(t-\tau))\right)\\
&= r_I(\tau)\cos\omega_c\tau.
\end{aligned}$$

(b) $X_{bb}(t) = X_I(t) + jX_Q(t)$.

$$\begin{aligned}
\Rightarrow \quad \mathbb{E}[X_{bb}(t)X_{bb}^*(t-\tau)] &= \mathbb{E}[X_I(t)X_I^*(t-\tau)] + \mathbb{E}[X_Q(t)X_Q^*(t-\tau)]\\
&= r_I(\tau) + r_Q(\tau)\\
&= 2r_I(\tau).
\end{aligned}$$

$$X_A(t) = X_{bb}(t)e^{j\omega_c t} \quad \Rightarrow \quad \mathbb{E}[X_A(t)X_A^*(t-\tau)] = \mathbb{E}[X_{bb}(t)X_{bb}^*(t-\tau)]e^{j\omega_c\tau}.$$

Hence

$$r_A(\tau) = 2r_I(\tau)e^{j\omega_c\tau}.$$

(c)

$$\mathbb{E}[y(k)y^*(k-l)] = 2r_I(l) = 2e^{-2|l|}.$$

$$\begin{aligned}
S_Y(D) &= 2\sum_{l=0}^{\infty} e^{-2l}D^l + 2\sum_{l=-\infty}^{-1} e^{2l}D^l \\
&= \frac{2}{1-e^{-2}D} + \frac{2e^{-2}D^{-1}}{1-e^{-2}D^{-1}} \\
&= 2\left(\frac{1-e^{-2}D^{-1}+e^{-2}D^{-1}(1-e^{-2}D)}{(1-e^{-2}D)(1-e^{-2}D^{-1})}\right) \\
&= 2\left(\frac{1-e^{-2}D^{-1}+e^{-2}D^{-1}-e^{-4}}{(1-e^{-2}D)(1-e^{-2}D^{-1})}\right) \\
&= \underbrace{2(1-e^{-4})}_{\gamma_0}\underbrace{\frac{1}{(1-e^{-2}D)}}_{L(D)}\underbrace{\frac{1}{(1-e^{-2}D^{-1})}}_{L^*(D^{-*})}
\end{aligned}$$

For example

$$W(D) = \frac{1-e^{-2}D^{-1}}{\sqrt{2(1-e^{-4})}}.$$

gives a whitening filter in baseband.

Problem 3.

- Computation of $Q(D)$

$$\begin{aligned}
\sum_{k:\text{odd}} 2^{-\frac{|k|-1}{2}}D^k &= \sum_{k:\text{odd},k>0} 2^{-\frac{|k|-1}{2}}D^k + \sum_{k:\text{odd},k<0} 2^{-\frac{|k|-1}{2}}D^k \\
& \quad k:\text{odd}, k<0 = \{-2m-1|m=0,1,...\} \Longrightarrow \frac{|k|-1}{2} = m \\
& \quad k:\text{odd}, k>0 = \{2n+1|n=0,1,...\} \Longrightarrow \frac{|k|-1}{2} = n \\
&= \sum_{n=0}^{\infty} 2^{-n}D^{2n+1} + \sum_{m=0}^{\infty} 2^{-m}D^{-2m-1} \\
&= D\sum_{n=0}^{\infty}(\frac{D^2}{2})^n + \frac{1}{D}\sum_{m=0}^{\infty}(\frac{1}{2D^2})^m \\
&= \frac{D}{1-\frac{D^2}{2}} + \frac{1}{D(1-\frac{1}{2D^2})} \\
&= \frac{D}{1-\frac{D^2}{2}} + \frac{D^{-1}}{1-\frac{D^{-2}}{2}}
\end{aligned}$$

$$\sum_{k:\text{even}} 2^{-\frac{|k|}{2}} D^k = \sum_{k:\text{even},k\geq 0} 2^{-\frac{|k|}{2}} D^k + \sum_{k:\text{even},k<0} 2^{-\frac{|k|}{2}} D^k$$

$$k:\text{even}, k<0 = \{-2m|m=1,2,...\} \Longrightarrow \frac{|k|}{2} = m$$

$$k:\text{even}, k\geq 0 = \{2n|n=0,1,...\} \Longrightarrow \frac{|k|}{2} = n$$

$$= \sum_{n=0}^{\infty} 2^{-n} D^{2n} + \sum_{m=1}^{\infty} 2^{-m} D^{-2m}$$

$$= \sum_{n=0}^{\infty} (\frac{D^2}{2})^n + \sum_{m=1}^{\infty} (\frac{1}{2D^2})^m$$

$$= \frac{1}{1-\frac{D^2}{2}} + \frac{\frac{1}{2D^2}}{1-\frac{1}{2D^2}}$$

$$= \frac{1}{1-\frac{D^2}{2}} + \frac{\frac{D^{-2}}{2}}{1-\frac{D^{-2}}{2}}$$

- Factorization of $Q(D)$

$$Q(D) = \sum_{k\in\mathbb{Z}} 2^{-\frac{|k|-1}{2}} D^k = \left[\frac{D}{1-\frac{D^2}{2}} + \frac{D^{-1}}{1-\frac{D^{-2}}{2}}\right] + \frac{5}{3}\left[\frac{1}{1-\frac{D^2}{2}} + \frac{\frac{D^{-2}}{2}}{1-\frac{D^{-2}}{2}}\right]$$

$$= \frac{\frac{5}{3}+D}{1-\frac{D^2}{2}} + \frac{\frac{5}{6D^2}+\frac{1}{D}}{1-\frac{1}{2D^2}}$$

$$= \frac{\frac{5}{3}-\frac{5}{6D^2}+D-\frac{1}{2D}+\frac{5}{6D^2}-\frac{5}{12}+\frac{1}{D}-\frac{D}{2}}{(1-\frac{1}{2}D^2)(1-\frac{1}{2}D^{-2})}$$

$$= \frac{\frac{5}{4}+\frac{1}{2}D+\frac{1}{2}D^{-1}}{(1-\frac{1}{2}D^2)(1-\frac{1}{2}D^{-2})}$$

$$= \frac{(1+\frac{1}{2}D)(1+\frac{1}{2}D^{-1})}{(1-\frac{1}{2}D^2)(1-\frac{1}{2}D^{-2})}$$

$$= F(D)F^*(D^{-*})$$

- Whitening Filter $W(D) = \frac{\sqrt{N_0}}{F^*(D^{-1*})}$
  We design $W(D)$ such that the $Q(D)W(D)$ and also $W(D)$ are stable & causal. So, $F^*(D^{-1*}) = \frac{1+\frac{1}{2}D}{1-\frac{1}{2}D^{-2}}$
  $W(D) = \sqrt{N_0}\frac{1-\frac{1}{2}D^{-2}}{1+\frac{1}{2}D}$
  In this case, both $Q(D)W(D)$ (filtered sequence) and $W(D)$ (whitening filter) are stable and casual.

Problem 4.

(a)

$$\psi(t) = \varphi(t) * h(t)$$

$$= \frac{1}{\sqrt{T}}\text{sinc}\left(\frac{t}{T}\right) * (\delta(t) - \frac{2}{5}\delta(t-T))$$

$$= \frac{1}{\sqrt{T}}\left(\text{sinc}\left(\frac{t}{T}\right) - \frac{2}{5}\text{sinc}\left(\frac{t}{T}-1\right)\right)$$

(b)

$$\begin{aligned}
||\psi||^2 &= \int_{-\infty}^{\infty} \psi^2(t)dt \\
&= \frac{1}{T}\left(\int_{-\infty}^{\infty} \operatorname{sinc} 2\left(\frac{t}{T}\right) - \frac{4}{5}\operatorname{sinc}\left(\frac{t}{T}\right)\operatorname{sinc}\left(\frac{t}{T}-1\right) + \frac{4}{25}\operatorname{sinc} 2\left(\frac{t}{T}-1\right)dt\right) \\
&= \frac{1}{T}\left(T + \frac{4}{25}T\right) = \frac{29}{25}
\end{aligned}$$

$$\tilde{\varphi}(t) = \frac{\psi(t)}{||\psi||} = \sqrt{\frac{25}{29T}}\left(\operatorname{sinc}\left(\frac{t}{T}\right) - \frac{2}{5}\operatorname{sinc}\left(\frac{t}{T}-1\right)\right)$$

(c) We have $\mathbb{E}[Z(t)Z^*(\tau)] = N_0$ and $Z_n =< Z, \tilde{\varphi}_n >$. So we can compute

$$\begin{aligned}
\mathbb{E}[z_n, z_{n-k}^*] &= \mathbb{E}\left[\left\{\int Z(t)\tilde{\varphi}_n(t)dt\right\}\left\{\int Z^*(\tau)\tilde{\varphi}_{n-k}^*(\tau)d\tau\right\}\right] \\
&= \int\int \tilde{\varphi}_n(t)\tilde{\varphi}_{n-k}^*(\tau)\mathbb{E}[Z(t)Z^*(\tau)]dtd\tau \\
&= \int\int N_0\delta(t-\tau)\tilde{\varphi}_n(t)\tilde{\varphi}_{n-k}^*(\tau)dtd\tau \\
&= N_0\int \tilde{\varphi}_n(t)\tilde{\varphi}_{n-k}^*(t)dt \\
&= N_0\int \tilde{\varphi}(t-nT)\tilde{\varphi}^*(t-(n-k)T)dt \\
&= \frac{25N_0}{29T}\int\Bigg[\operatorname{sinc}\left(\frac{t}{T}-n\right)\operatorname{sinc}\left(\frac{t}{T}-(n-k)\right) \\
&\quad -\frac{2}{5}\operatorname{sinc}\left(\frac{t}{T}-(n+1)\right)\operatorname{sinc}\left(\frac{t}{T}-(n-k)\right) \\
&\quad -\frac{2}{5}\operatorname{sinc}\left(\frac{t}{T}-n\right)\operatorname{sinc}\left(\frac{t}{T}-(n-k+1)\right) \\
&\quad +\frac{4}{25}\operatorname{sinc}\left(\frac{t}{T}-(n+1)\right)\operatorname{sinc}\left(\frac{t}{T}-(n-k+1)\right)\Bigg]dt \\
\Rightarrow \quad & \mathbb{E}[z_n, z_{n-k}^*] = N_0 q_k \\
& \text{where } q_{-1} = -\frac{10}{29}, q_0 = 1, q_1 = -\frac{10}{29}
\end{aligned}$$

So $S_Z(D) = N_0 Q(D)$, where $Q(D)$ is the D-Transform of q_k.
The equivalent channel in the D domain is

$$Y(D) = \frac{1}{N_0}S_Z(D)||p||X(D) + Z(D)$$

To obtain a white noise, we must find a filter $G(D)$ such that the power spectral density of the noise is constant. In order to find such a filter, let's take the spectral factorization of $S_Z(D)$.

$$S_Z(D) = N_0 Q(D) = N_0(1 - \frac{10}{29}D - \frac{10}{29}D^{-1}) = F(D)F^*(D^{-*})$$

$$S_Z(D) = \sqrt{\frac{25N_0}{29}}(1 - \frac{2}{5}D)\sqrt{\frac{25N_0}{29}}(1 - \frac{2}{5}D^{-1})$$

And $F(D) = \sqrt{\frac{25N_0}{29}}(1 - \frac{2}{5}D)$. The whitening filter $G(D)$ is $\frac{\sqrt{N_0}}{F^*(D^{-*})} = \sqrt{\frac{29}{25}}\frac{1}{(1-\frac{2}{5}D^{-1})}$. The resulting channel is

$$\tilde{Y}(D) = Y(D)G(D) = \frac{1}{\sqrt{N_0}}F(D)||p||X(D) + W(D)$$

where $W(D) = \frac{Z(D)\sqrt{N_0}}{F^*(D^{-*})}$ is white noise and $S_W(D) = N_0$. So

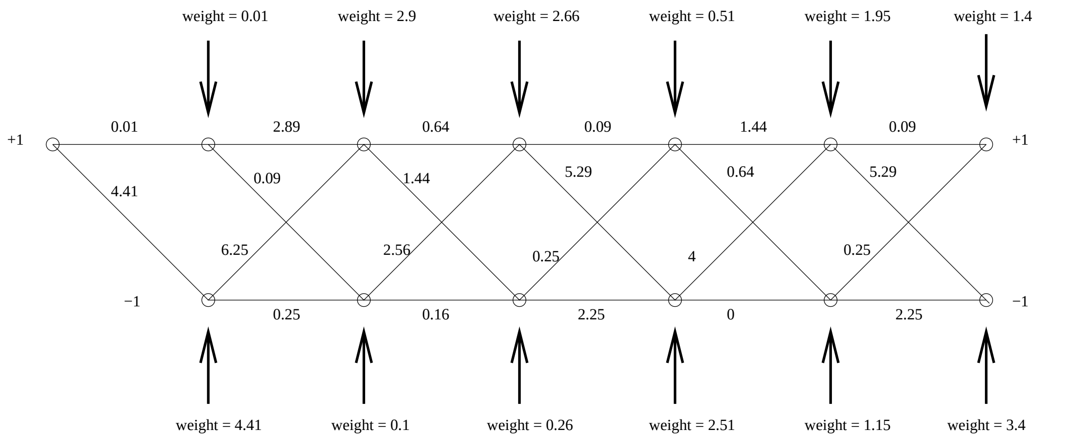
$$\tilde{Y}(D) = \sqrt{\frac{25}{29}}(1 - \frac{2}{5}D)\sqrt{\frac{29}{25}}X(D) + W(D)$$

$$\tilde{Y}(D) = (1 - \frac{2}{5}D)X(D) + W(D).$$

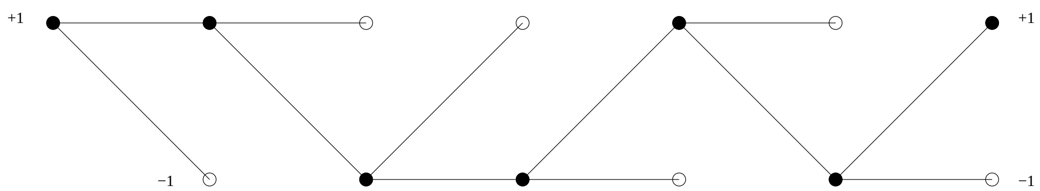
(d) The resulting channel in the time domain is

$$\tilde{y}_k = x_k - \frac{2}{5}x_{k-1} + w_k$$

The channel in the problem has a state of just X_{k-1} as $\nu = 1$. The Viterbi algorithm assumes a channel of the following form $y_k = u_k + z_k$ where $u_k = f_k * x_k$. In the considered channel $u_k = x_k - \frac{2}{5}x_{k-1}$ and $f_0 = 1$ and $f_1 = -\frac{2}{5}$.
Recall that the metric to use is $|y_k - u_k|^2$, and the maximum likelihood detection criterion is $\{x_k\} = \arg\min \sum_k |y_k - u_k|^2$
Shown below is the Viterbi algorithm decoding trellis.



Pruning the trellis gives us the sequence estimates. We used the fact that the initial and the end states are 1. Note that the choosing the end state to be 1 is a natural choice, because its accumulated weight is 1.4 and therefore smaller than the weight (3.4) of the -1 end state.



Hence the estimated input sequence is $\widehat{\mathbf{X}} = [+1, -1, -1, +1, -1, +1]$.