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Handout 6 Advanced Digital Communications
Homework 4 October 15, 2010

PROBLEM 1. Suppose Z is a complex random variable with density p_Z.

(a) Let $R = |Z|$. Show that the density p_R of R is given by

$$p_R(r) = r \int_0^{2\pi} p_Z(r \exp(j\theta))\, d\theta.$$

Hint: Write $\Pr(R \leq r)$ as an integral over x and y, then use polar coordinates.

(b) Let $U = R^2$. Show that its density is given by

$$p_U(u) = \frac{1}{2} \int_0^{2\pi} p_Z(\sqrt{u} \exp(i\theta))\, d\theta.$$

(c) Suppose now that Z is circularly symmetric. Show that

$$p_U(u) = \pi p_Z(\sqrt{u}).$$

(d) Again suppose Z is circularly symmetric. Let X and Y be its real imaginary parts. We know that X and Y are identically distributed, call the common density p. Suppose that X and Y are independent. Show that

$$p_U(x^2 + y^2) = \pi p(x)p(y).$$

(e) Under the assumptions of (d), conclude that

$$p_U(x^2 + y^2) = \frac{1}{\pi p(0)^2} p_U(x^2) p_U(y^2).$$

Assuming that p_U is continuous show that it must be given by

$$p_U(u) = \alpha \exp(-\alpha u), \quad u \geq 0,$$

where $\alpha = \pi p(0)^2$. Hint: The only continuous functions f that satisfies $f(a + b) = f(a)f(b)$ are those for which $f(a) = \exp(\beta a)$ for some β.

(f) Show that if Z is circularly symmetric complex random variable with independent real and imaginary parts, then Z must be Gaussian.

PROBLEM 2. Let $\mathbf{Z} = (Z_1, \ldots, Z_n)^T$ be a vector of complex iid Gaussian rvs with iid real and imaginary parts, each $N(0, \frac{N_0}{2})$. The input $\mathbf{U}$ is binary antipodal, taking on values $\mathbf{a}$ or $-\mathbf{a}$, where $\mathbf{a} = (a_1, \ldots, a_n)^T$ is an arbitrary complex n-vector. The observation $\mathbf{V}$ is $\mathbf{U} + \mathbf{Z}$, and the probability density of $\mathbf{Z}$ is given by

$$f_Z(z) = \frac{1}{(\pi N_0)^n} e^{(\sum_{j=1}^n \frac{-|z_j|^2}{N_0})} = \frac{1}{(\pi N_0)^n} e^{\frac{-||Z||^2}{N_0}}.$$

(a) Give expressions for $f_{V|U}(\mathbf{v}|a)$ and $f_{V|U}(\mathbf{v}|-a)$.

(b) Show that the log likelihood ratio for the observation $\mathbf{v}$ is given by

$$\mathrm{LLR}(\mathbf{v}) = \frac{-||\mathbf{v}-\mathbf{a}||^2 + ||\mathbf{v}+\mathbf{a}||^2}{N_0}.$$

(c) Explain why this implies that ML detection is minimum distance detection (defining the distance between two complex vectors as the norm of their difference).

(d) Show that $\mathrm{LLR}(\mathbf{v})$ can also be written as $\frac{4\mathrm{Re}(\langle \mathbf{v},\mathbf{a}\rangle)}{N_0}$.

(e) The appearance of the real part, $\mathrm{Re}(\langle \mathbf{v},\mathbf{a}\rangle)$, in part (d) is surprising. Point out why log likelihood ratios must be real. Also explain why replacing $\mathrm{Re}(\langle \mathbf{v},\mathbf{a}\rangle)$ by $|\langle \mathbf{v},\mathbf{a}\rangle|$ in the above expression would give a non-sensical result in the ML test.

(f) Does the set of points $\{\mathbf{v} : \mathrm{LLR}(\mathbf{v}) = 0\}$ form a complex vector space?

Problem 3. (Amplitude-limited functions) Sometimes it is important to generate baseband waveforms with bounded amplitude. This problem explores pulse shapes that can accomplish this.

(a) Find the Fourier transform of $g(t) = \mathrm{sinc}^2(2Wt)$. Show that $g(t)$ is bandlimited to $f \leq W$ and sketch both $g(t)$ and $\hat{g}(f)$. [*Hint.* Recall that multiplication in the time domain corresponds to convolution in the frequency domain.]

(b) Let $u(t)$ be a continuous real $\mathcal{L}_2$ function baseband-limited to $f \leq W$ (i.e. a function such that $u(t) = \sum_k u(kT)\mathrm{sinc}(\frac{t}{T} - k)$, where $T = \frac{1}{2W}$. Let $v(t) = u(t) * g(t)$. Express $v(t)$ in terms of the samples $\{u(kT); k \in \mathcal{Z}\}$ of $u(t)$ and the shifts $\{g(t - kT); k \in \mathcal{Z}\}$ of $g(t)$. [*Hint.* Use your sketches in part (a) to evaluate $g(t) * \mathrm{sinc}(\frac{t}{T})$.]

(c) Show that if the T-spaced samples of $u(t)$ are nonnegative, then $v(t) \geq 0$ for all t.

(d) Explain why $\sum_k \mathrm{sinc}(\frac{t}{T} - k) = 1$ for all t.

(e) Using (d), show that $\sum_k g(\frac{t}{T} - k) = c$ for all t and find the constant c. [*Hint.* Use the hint in (b) again.]

(f) Now assume that $u(t)$, as defined in part (b), also satisfies $u(kT) \leq 1$ for all $k \in \mathcal{Z}$. Show that $v(t) \leq 2$ for all t.

(g) Allow $u(t)$ to be complex now, with $|u(kT)| \leq 1$. Show that $v(t) \leq 2$ for all t.

Problem 4. (Orthogonal sets) The function $\mathrm{rect}(\frac{t}{T})$ has the very special property that it, plus its time and frequency shifts, by kT and $\frac{j}{T}$, respectively, form an orthogonal set. The function $\mathrm{sinc}(\frac{t}{T})$ has the same property. We explore other functions that are generalizations of $\mathrm{rect}(\frac{t}{T})$ and which, as you will show in parts (a)-(d), have this same interesting property. For simplicity, choose $T = 1$. These functions take only the values 0 and 1 and are allowed to be nonzero only over $[-1; 1]$ rather than $[-\frac{1}{2}, \frac{1}{2}]$ as with $\mathrm{rect}(\frac{t}{T})$. Explicitly, the functions considered here satisfy the following constraints:

$$\begin{array}{rcc}
p(t) = & p^2(t) & \text{for all } t \text{ (0/1 property);} \\
p(t) = & 0 & \text{for } |t| > 1; \\
p(t) = & p(-t) & \text{for all } t \text{ (symmetry);} \\
p(t) = & 1 - p(t-1) & \text{for } 0 \leq t \leq 1/2.
\end{array}$$

Note: because of property (3), condition (4) also holds for $1/2 < t \le 1$. Note also that $p(t)$ at the single points $t = \pm\frac{1}{2}$ does not affects any orthogonality properties, so you are free to ignore these points in your arguments.

(a) Show that $p(t)$ is orthogonal to $p(t-1)$.

Hint. Evaluate $p(t)p(t-1)$ for each $t \in [0;1]$ other than $t = \frac{1}{2}$.

(b) Show that $p(t)$ is orthogonal to $p(t-k)$ for all integer $k \neq 0$.

(c) Show that $p(t)$ is orthogonal to $p(t-k)e^{j2\pi mt}$ for integer $k \neq 0$ and $m \neq 0$.

(d) Show that $p(t)$ is orthogonal to $p(t)e^{j2\pi mt}$ for integer $m \neq 0$.

Hint. Evaluate $p(t)e^{j2\pi mt} + p(t-1)e^{j2\pi m(t-1)}$.

(e) Let $h(t) = \hat{p}(t)$ where $\hat{p}(f)$ is the Fourier transform of $p(t)$. If $p(t)$ satisfies properties (1)-(4), does it follow that $h(t)$ has the property that it is orthogonal to $h(t-k)e^{j2\pi mt}$ whenever either the integer k or m is nonzero?

Note: almost no calculation is required in this problem.