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School of Computer and Communication Sciences

Handout 5 Advanced Digital Communications
Homework 3 October 11, 2009

Problem 1.

(a) As discussed in HW1, Problem 2, the Markov bound on the probability that a real random variable Z exceeds b is given by

$$\Pr(Z \geq b) \leq \frac{E(Z)}{b}.$$

Use Markov bound to derive the Chernoff bound on the probability that a real random variable Z exceeds b is given by

$$\Pr(Z \geq b) \leq E\big(e^{s(Z-b)}\big), \qquad s \geq 0.$$

Hint. $e^{s(z-b)} \geq 1$ when $z \geq b$, and $e^{s(z-b)} \geq 0$ otherwise.

(b) Use the Chernoff bound to show that

$$Q(x) \leq e^{-\frac{x^2}{2}}$$

(c) Integrate by parts to derive the upper and lower bounds

$$Q(x) \leq \frac{1}{\sqrt{2\pi x^2}} e^{-\frac{x^2}{2}} \tag{1}$$

$$Q(x) \geq \big(1 - \frac{1}{x^2}\big)\frac{1}{\sqrt{2\pi x^2}} e^{-\frac{x^2}{2}}. \tag{2}$$

(d) Here is another way to establish these tight upper and lower bounds. By using a simple change of variables, show that

$$Q(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \int_0^\infty e^{\frac{-y^2}{2} - xy}\, dy$$

Then show that

$$1 - \frac{y^2}{2} \leq e^{\frac{-y^2}{2}} \leq 1$$

Putting these together, derive the bounds of part (c).

For (e)-(g), consider a circle of radius x inscribed in a square of side $2x$.

(e) Show that the probability that a two-dimensional iid real Gaussian random variable X with variance $\sigma^2 = 1$ per dimension falls inside the square is equal to $(1-2Q(x))^2$.

(f) Show that the probability that X falls inside the circle is $1 - e^{-\frac{x^2}{2}}$.

Hint. Write $p_X(x)$ in polar coordinates.

(g) Show that (e) and (f) imply that when x is large,

$$Q(x) \leq \frac{1}{4} e^{-\frac{x^2}{2}}.$$

Problem 2. An orthogonal signal constellation is a set $A = \{a_j, 1 \leq j \leq M\}$ of M orthogonal vectors in $\mathcal{R}^M$ with equal energy $\mathcal{E}$; i.e., $\langle a_j, a_i \rangle = \mathcal{E}\delta_{ji}$.

(a) Compute the nominal spectral efficiency ρ of A in bits per dimension. Compute the average energy E_b per information bit.

Hint. In a N-dimensional M-point constellation, we define,

- The bit rate (nominal spectral efficiency) $\rho = \frac{1}{N} \log_2 M$ b/D
- The average energy per dimension $E_s = \frac{1}{N}\mathcal{E}$, or the average energy per bit $E_b = \frac{\mathcal{E}}{\log_2 M} = \frac{E_s}{\rho}$.

(b) Compute the minimum squared distance $d^2_{\min}(A)$. Show that every signal has $M-1$ nearest neighbors.

(c) Let the noise variance be σ^2 per dimension. Show that the probability of error of an optimum detector is bounded by the union bound error

$$\Pr(E) \leq (M-1)Q\Big(\sqrt{\frac{2\mathcal{E}}{\sigma^2}}\Big).$$

(d) Let $M \to \infty$ with E_b held constant. Using an asymptotically accurate upper bound for Q function, show that $\Pr(E) \to 0$ provided that $\frac{E_b}{\sigma^2} > \ln 2$. What is the nominal spectral efficiency ρ in the limit?

Hint.

$$\begin{aligned} Q(x) &\leq \frac{1}{\sqrt{2\pi x^2}} e^{-\frac{x^2}{2}} \\ Q(x) &\geq \big(1 - \frac{1}{x^2}\big)\frac{1}{\sqrt{2\pi x^2}} e^{-\frac{x^2}{2}}. \end{aligned}$$

Assume that we use the following decoder: if there exists exactly one a_i such that $\langle a_i, y \rangle \geq T$ then decide i otherwise declare an error. $T \in \mathcal{R}$ is a fixed threshold. As the maximum likelihood decoder minimizes the error probability among all decoders, any upper bound to the error probability of this suboptimal decoder is an upper bound to the error probability of the ML decoder.

(e) Compute the distribution of $\Pr(\langle a_i, Y \rangle \leq T | X = a_i)$ and $\Pr(\langle a_i, Y \rangle \geq T | X = a_j)$ with $j \neq i$.

(f) Let $\Pr(E)$ the probability of error of the decoder. Prove the following:

$$\Pr(E) \leq Q\left(\frac{\mathcal{E} - T}{\sigma \cdot \sqrt{\mathcal{E}}}\right) + (M-1)Q\left(\frac{T}{\sigma\sqrt{\mathcal{E}}}\right)$$

(g) Let $M \to \infty$ with E_b held constant and set $T = \alpha \cdot \mathcal{E}$ with $0 \leq \alpha < 1$. Proved that $\Pr(E) \to 0$ provided that $\alpha^2 E_b/\sigma^2 > 2\ln 2$ conclude that when $E_b/\sigma^2 > 2\ln 2$ we can find a decoder for which $P(E) \to 0$ as $M \to \infty$.

Problem 3. Let the transmitted bandpass signal be given by

$$x(t) = a\cos\big(2\pi(f_c + \frac{1}{T})t\big) + b\cos\big(2\pi(f_c + \frac{2}{T})t\big), \qquad t \in [0, T]$$

and $a \in \{0, A\}$, $b \in \{0, A\}$.

1. Find the baseband equivalent signal $x_{bb}(t)$ for the transmitted signal.

2. Find the vector representation of the baseband signal and draw the corresponding signal constellation.

3. If $a = \begin{cases} 0 & \text{w.p.} \frac{1}{2} \\ A & \text{w.p.} \frac{1}{2} \end{cases}$ and $b = \begin{cases} 0 & \text{w.p.} \frac{1}{2} \\ A & \text{w.p.} \frac{1}{2} \end{cases}$.

   Find the average energy of the baseband signal. Is this a minimum energy configuration? If not how will you modify the constellation so that it is of minimum energy?

Problem 4. Consider a PAM baseband system in which the modulator is defined by a signal interval T and a waveform $p(t)$, the channel is defined by a filter $h(t)$, and the receiver is defined by a filter $q(t)$ which is sampled at T-spaced intervals. The received waveform, after the receiver filter $q(t)$, is then given by $r(t) = \sum_k u_k g(t - kT)$, where $g(t) = p(t) * h(t) * q(t)$.

(a) What property must $g(t)$ have so that $r(kT) = u_k$ for all k and for all choices of input $\{u_k\}$? What is the Nyquist criterion for $\hat{g}(f)$?

(b) Now assume that $T = \frac{1}{2}$ and that $p(t), h(t), q(t)$ and all their Fourier transforms are restricted to be real. Assume further that $\hat{p}(f)$ and $\hat{h}(f)$ are specified by

$$\hat{p}(f) = \begin{cases} 1 & |f| \le 0.5 \\ 1.5 - |f| & 0.5 \le |f| \le 1.5 \\ 0 & |f| > 0 \end{cases}$$

and

$$\hat{h}(f) = \begin{cases} 1 & |f| \le 0.75 \\ 0 & 0.75 \le |f| \le 1 \\ 1 & 1 \le |f| \le 1.25 \\ 0 & |f| > 0 \end{cases}.$$

Is it possible to choose a receiver filter transform $\hat{q}(f)$ so that there is no intersymbol interference? If so, give such a $\hat{q}(f)$ and indicate the regions in which your solution is nonunique.

(c) Redo part (b) with the modification that now $\hat{h}(f) = 1$ for $|f| \le 0.75$ and $\hat{h}(f) = 0$ for $|f| > 0.75$.

(d) Explain the conditions on $\hat{p}(f)\,\hat{h}(f)$ under which intersymbol interference can be avoided by proper choice of $\hat{q}(f)$. (You may assume, as above, that $\hat{p}(f)$, $\hat{h}(f)$, $p(t)$ and $h(t)$ are all real.)