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School of Computer and Communication Sciences

**Handout 7** Advanced Digital Communications
Homework 2 (Solutions) October 18, 2010

PROBLEM 1.

$$R(H) = \int_Y \sum_j \sum_i \pi_j C_{i;j} \Pr(Y \in \Gamma_i | H_j) \quad (1)$$

$$= \sum_i \int_{Y \in \Gamma_i} \sum_j \pi_j C_{i;j} \Pr(Y \in \Gamma_i | H_j) \quad (2)$$

Now suppose, the decoder wants to associate $Y = y$ to one of the decision regions $\Gamma_i$ so that the risk $R(H)$ is minimized. Therefore, the decoder chooses $y$ to be in $\Gamma_i$, in which $y$ minimizes $\sum_j \pi_j C_{i;j} \Pr(y|H_j)$

$$Y \in \Gamma_i : i = \underset{i}{\operatorname{argmin}} \sum_j \pi_j C_{i;j} \cdot \Pr(y|H_j)$$

For the binary case the problem of finding the minimum is turned into a simple inequality checking. Therefore, we have

$$\sum_{j=0}^{1} \pi_j C_{0;j} \Pr(y|H_j) \underset{H_0}{\overset{H_1}{\gtrless}} \sum_{j=0}^{1} \pi_j C_{1;j} \Pr(y|H_j) \quad (3)$$

$$\pi_0 C_{0;0} \Pr(y|H_0) + \pi_1 C_{0;1} \Pr(y|H_1) \underset{H_0}{\overset{H_1}{\gtrless}} \pi_0 C_{1;0} \Pr(y|H_0) + \pi_1 C_{1;1} \Pr(y|H_1) \quad (4)$$

$$(\pi_1 C_{0;1} - \pi_1 C_{1;1}) \Pr(y|H_1) \underset{H_0}{\overset{H_1}{\gtrless}} (\pi_0 C_{1;0} - \pi_0 C_{0;0}) \Pr(y|H_0) \quad (5)$$

$$\frac{\Pr(y|H_1)}{\Pr(y|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\pi_0 (C_{1;0} - C_{0;0})}{\pi_1 (C_{0;1} - C_{1;1})} \quad (6)$$

The decision making only depends on the ratio $\Pr(y|H_1)/\Pr(y|H_0)$ and not the individual values of $\Pr(y|H_1)$ and $\Pr(y|H_0)$, and likelihood ratio is a sufficient statistics for optimal decision rule.

Now we have:

$$\frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(y-1)^2}{2\sigma^2}) \pi_1 \gamma \underset{H_0}{\overset{H_1}{\gtrless}} \frac{1}{\sqrt{2\pi}} \exp(-\frac{y^2}{2}) \pi_0 \quad (7)$$

Clearly, if $\gamma$ goes to infinity, for any given value of $\pi_1 \neq 0$, and $y$ and $\sigma^2$ finite, decoder chooses $H_1$.

We will have the following decision regions:

$$\frac{(y-1)^2}{2\sigma^2} - \frac{y^2}{2} \underset{H_1}{\overset{H_0}{\gtrless}} \ln \frac{\pi_1 \gamma}{\pi_0 \sigma} \quad (8)$$

$$(1-\sigma^2) y^2 - 2y + 1 \underset{H_1}{\overset{H_0}{\gtrless}} 2\sigma^2 \ln \frac{\pi_1 \gamma}{\pi_0 \sigma} \quad (9)$$

Problem 2.

(a) Because of the additive nature of the channel, the error probability only depends on $P_N(\overrightarrow{y} - \overrightarrow{x_i})$. Now, if one shifts all $\overrightarrow{x_i}$'s by a constant vector, by shifting all decision regions by the same constant vector, one will design an equal error probability system for the second signal set.

(b)

$$A' = A - m(a) = \{a_j - m(a), \quad 1 \leq j \leq M\} \tag{10}$$

$$E(A') = \frac{1}{M}\sum_j \langle (a_j - m(a)), (a_j - m(a)) \rangle \tag{11}$$

$$= \frac{1}{M}\sum_j \langle a_j, a_j \rangle + \frac{1}{M}\sum_j \langle m(a), m(a) \rangle - \frac{2}{M}\sum_j \langle a_j, m(a) \rangle \tag{12}$$

$$= \frac{1}{M}\sum_j \langle a_j, a_j \rangle + \langle m(a), m(a) \rangle - 2\langle \frac{1}{M}\sum_j a_j, m(a) \rangle \tag{13}$$

$$= \frac{1}{M}\sum_j \langle a_j, a_j \rangle - \langle m(a), m(a) \rangle \tag{14}$$

$$= E(A) - \langle m(a), m(a) \rangle \tag{15}$$

By part (a), adding a constant vector $(-m(A))$ does not change the error probability, but it reduces the average transmitted energy, so it is good.

Problem 3.

(a) $V(R)$ for n-cube is $(2M)^n$, so number of signal points is $\frac{(2M)^n}{2^n} = M^n$.

$$E(R) = \int_R ||x||^2 P(x)dx = \int_{-M}^{M} ... \int_{-M}^{M} \sum_{i=1}^{n} x_i^2 \frac{1}{(2M)^n} dx_1...dx_n \tag{16}$$

$$= \sum_{i=1}^{n} \int_{-M}^{M} ... \int_{-M}^{M} x_i^2 \frac{1}{(2M)^n} dx_1...dx_n \tag{17}$$

$$= \sum_{i=1}^{n} \int_{-M}^{M} x_i^2 \frac{(2M)^{n-1}}{(2M)^n} dx_1 \tag{18}$$

$$= \sum_{i=1}^{n} \frac{1}{2M} \int_{-M}^{M} x_1^2 dx_1 \tag{19}$$

$$= n\frac{1}{2M}\frac{2M^3}{3} = n\frac{M^2}{3} \tag{20}$$

They are exact because a $n$-cube constellation of size $2M$ is the $n$-fold Cartesian product of an M-PAM constellation of the set of all odd integers in the interval [-M,M].

(b)

$$\text{Number of points :} \quad \frac{(\pi r^2)^{\frac{n}{2}}}{(\frac{n}{2})! 2^n}$$

$$\text{Average energy :} \quad \frac{nr^2}{n+2}$$

(c) For $n = 2$ and same number of signal points, we have:

$$M^2 = \frac{\pi r^2}{4} \tag{21}$$
$$\frac{r^2}{M^2} = \frac{4}{\pi} \tag{22}$$

So

$$\frac{E_{\text{sphere}}}{E_{\text{cube}}} = \frac{\frac{r^2}{2}}{\frac{2}{3}M^2} \tag{23}$$
$$= \frac{r^2}{M^2} \cdot \frac{3}{4} = \frac{3}{\pi} = -0.2dB \tag{24}$$

(d)

$$M^{16} = \frac{(\pi r^2)^8}{8!2^16} \tag{25}$$
$$\Rightarrow M^2 = \frac{\pi r^2}{(8!)^{\frac{1}{8}}4} \tag{26}$$
$$\frac{E_{\text{sphere}}}{E_{\text{cube}}} = \frac{\frac{\cancel{16}r^2}{18}}{\frac{\cancel{16}M^2}{3}} \tag{27}$$
$$= \frac{r^2}{6M^2} = \frac{(8!)^{\frac{1}{8}} \cdot 4}{\pi \cdot 6} \approx -1dB \tag{28}$$

(e) We have

$$M^n = \frac{(\pi r^2)^{\frac{n}{2}}}{\frac{n}{2}!2^n} \tag{29}$$
$$\Rightarrow M^2 = \frac{\pi r^2}{4((\frac{n}{2}!))^{\frac{2}{n}}} \tag{30}$$
$$\tag{31}$$

So

$$\frac{E_{\text{sphere}}}{E_{\text{cube}}} = \frac{\frac{n}{n+2}r^2}{n\frac{M^2}{3}} = \frac{3r^2}{(n+2)M^2} \tag{32}$$
$$= \frac{3}{n+2} \cdot \frac{4((\frac{n}{2})!)^{\frac{2}{n}}}{\pi} \tag{33}$$
$$= \frac{12}{(n+2)\pi} \cdot ((\frac{n}{2e})^{\frac{n}{2}})^{\frac{2}{n}} \cdot (\sqrt{2\pi\frac{n}{2}})^{\frac{2}{n}} \tag{34}$$
$$= \frac{6}{\pi e} \cdot \frac{n}{n+2}(\pi n)^{\frac{1}{n}} \tag{35}$$

$$\lim_{n\to\infty} \frac{E_{\text{sphere}}}{E_{\text{cube}}} = \frac{6}{\pi e} = -1.53dB$$

Problem 4. (a) Given the observation $(y_1, y_2)$, the maximum likelihood receiver computes for each hypothesis $x$

$$\text{score}(x) = p((y_1, y_2)|x) = p(y_1|x)p(y_2|y_1, x)$$

and chooses the $x$ with the highest score. If $p(y_2|y_1, x) = p(y_2|y_1)$, then

$$\text{score}(x) = p(y_1|x)p(y_2|y_1).$$

Since the factor $p(y_2|y_1)$ is common to the score of each $x$, the ranking of the $x$'s will not change if it is based on the modified score

$$\text{score}'(x) = p(y_1|x).$$

As score$'$ can be computed from $y_1$ alone, the receiver does not need $y_2$ to make its decision.

(b) (i). With $Y_1 = X + N_1$, $Y_2 = X + N_2$, $Y_3 = X + N_1 + N_2$ with independent $X, N_1, N_2$,

$$\begin{aligned}
\Pr(Y_3 \le y_3|Y_1 = y_1, X = x) &= \Pr(X + N_1 + N_2 \le y_3|Y_1 = y_1, X = x) \\
&= \Pr(N_2 \le y_3 - y_1|Y_1 = y_1, X = x) \\
&= \Pr(N_2 \le y_3 - y_1) \qquad (*) \\
&= \Pr(Y_3 \le y_3|Y_1 = y_1)
\end{aligned}$$

where (*) follows from the independence of $N_2$ from $X$ and $N_1$. Thus, $p(y_3|y_1, x) = p(y_3|y_1)$ and we conclude that $y_3$ is irrelevant given only $y_1$.

(ii). Given $Y_1$ and $Y_2$, the knowledge of $Y_3$ would let us determine $X$ exactly as $X = Y_1 + Y_2 - Y_3$. Such exact determination is in general not possible from $Y_1$ and $Y_2$ alone, so $Y_3$ is not irrelevant.

Under special circumstances the pair $Y_1, Y_2$ may determine $X$ exactly, and $Y_3$ is irrelevant. Some examples: (1) $X$ is a constant; (2) $N_1 = 0$ with probability 1; or perhaps more interestingly, (3) $X$ takes only values in $\{0, 1, 2, 3, 4, 5\}$, $N_1$ takes only values in even integers and $N_3$ is always a multiple of 3, then, from $Y_1$ we know ($X$ mod 2), from $Y_2$ we know ($X$ mod 3), so we can find ($X$ mod 6) and thus determine $X$.

(c) The conditional cumulative distribution of $Y_2$,

$$\Pr(Y_2 \le y_2|Y_1 = y_1, X = x) = \Pr(N_2 \le y_2 - x)$$

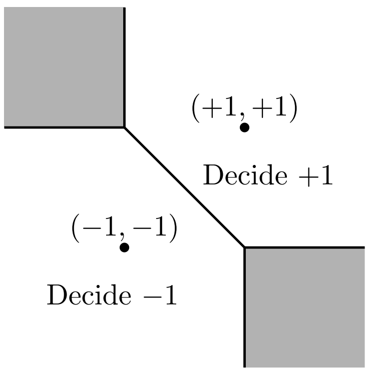
is a function that depends on the value of $x$. If $P(Y_2 \le y_2|Y_1 = y_1, X = x)$ were equal to $P(Y_2 \le y_2|Y_1 = y_1)$ this would not have been the case. So, $Y_2$ is not irrelevant.

(d) Observe that

$$\log P(y_1, y_2|x) = \log P_{N_1}(y_1 - x) + \log P_{N_2}(y_2 - x) = -\big[|y_1 - x| + |y_2 - x|\big] - \log 2.$$

Thus the optimum decision rule is

$$\begin{cases} +1 & |y_1 - 1| + |y_2 - 1| < |y_1 + 1| + |y_2 + 1| \\ -1 & |y_1 - 1| + |y_2 - 1| > |y_1 + 1| + |y_2 + 1| \\ \text{either} & |y_1 - 1| + |y_2 - 1| = |y_1 + 1| + |y_2 + 1| \end{cases} = \begin{cases} +1 & g(y_1) + g(y_2) > 0 \\ -1 & g(y_1) + g(y_2) < 0 \\ \text{either} & g(y_1) + g(y_2) = 0 \end{cases}$$




with

$$g(y) = |y+1| - |y-1| = \begin{cases} -2 & y < -1 \\ 2y & -1 \le y \le 1 \\ +2 & y > 1. \end{cases}$$

The decision regions are shown in the figure with the gray zones indicating the when the decision is arbitrary.

(e) Since the rule agrees with the rule derived in part (d) it is optimum for the case of equally likely messages. By symmetry, the probability of error can be computed as $P(\text{error}) = P(\text{error}|X=-1)$, with is the same as

$$\Pr(Y_1 + Y_2 \ge 0 | X = -1) = \Pr(N_1 + N_2 \ge 2).$$

Writing the above as

$$\int p_{N_1}(n_1) P(N_2 > 2 - n_1)\, dn_1,$$

observing that

$$P(N_2 > x) = \begin{cases} \exp(-x)/2 & x \ge 0 \\ 1 - \exp(x)/2, & x < 0, \end{cases}$$

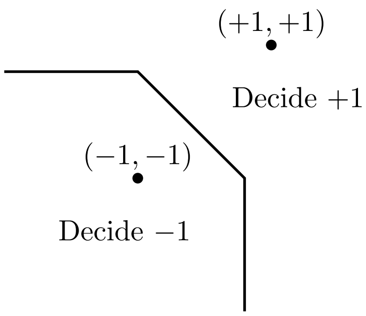
and substituting $p_{N_1}(x) = \exp(-|x|)/2$, we can compute the probability of error (above integration) as follows:

$$\int_{-\infty}^{+2} \frac{e^{-|n1|}}{2} \frac{e^{-2+n1}}{2} dn1 + \int_{+2}^{+\infty} \frac{e^{-|n1|}}{2} (1 - \frac{e^{2-n1}}{2}) dn1 = 1/e^2$$

(f) The MAP rule is given by decision $= \arg \max_{x \in \{+1,-1\}} P(y_1, y_2|x)p(x)$,

which, with $q = \Pr(X = +1)$, simplifies to

$$\begin{cases} +1 & g(y_1) + g(y_2) > \log((1-q)/q) \\ -1 & g(y_1) + g(y_2) < \log((1-q)/q) \\ \text{either} & g(y_1) + g(y_2) = \log((1-q)/q) \end{cases}$$




With $q > 1/2$, this has the effect of eliminating the gray zone, and shrinking the decision region for $X = -1$ as shown.

PROBLEM 5. 1. $[1, 1, 1, 1]$,
$[1, 1, -1, -1]$,

$[-1,-1,1,1]$,
$[-1,-1,-1,-1]$,
$[-1,1,-1,1]$,
$[-1,1,1,-1]$,
$[1,-1,1,-1]$,
$[1,-1,-1,1]$.

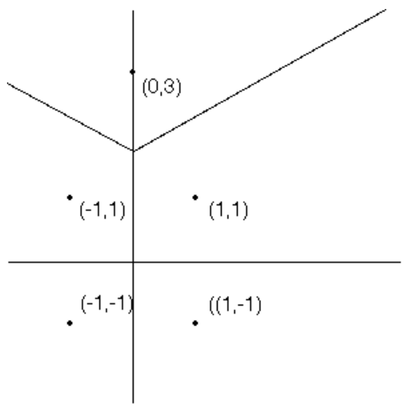
2.
$$b = \log_2 8 = 3$$
$$\bar{b} = \frac{3}{4}$$
.

3.
$$E_x = \frac{1}{8}\sum_{i=0}^{7} \|x_i\|^2 = 4$$

So $\overline{E_x} = 1$.

4. For each point, we can find 6 points at equal minimum distance $2\sqrt{2}$, so $N_e = 6$ and
$$P_e = N_e Q\left(\frac{d_{min}}{2\sigma}\right)$$
$$P_e = 6Q\left(\frac{2\sqrt{2}}{2\sqrt{0.1}}\right) = 6Q(\sqrt{20})$$

PROBLEM 6. 1. Here are the decision regions:




2. • Union bound with $d_{min} = 2$ and $N_i = 4$
$$P_e \le N_i Q(\frac{d_{min}}{2\sigma}) = 4Q(\frac{1}{\sigma})$$

• Nearest Neighbor Union Bound with $d_{min} = 2$ and $N_e = \frac{2+3+3+2+2}{5} = \frac{12}{5}$
$$P_e \le N_e Q(\frac{d_{min}}{2\sigma}) = \frac{12}{5} Q(\frac{1}{\sigma})$$

Please note that you can get better (tighter) bounds if you use the exact distances between neighboring points instead of $d_{min}$.