ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 3 — Advanced Digital Communications

Homework 2 — October 1, 2010

PROBLEM 1. Consider the simple hypothesis testing problem for the real-valued observation Y :

$$H_0 : p_0(y) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{y^2}{2}); y \in \mathbf{R} \tag{1}$$

$$H_1 : p_1(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(y-1)^2}{2\sigma^2}); y \in \mathbf{R} \tag{2}$$

Let $C_{i;j}$ denote the cost of choosing hypothesis i when actually hypothesis j was true. Then the expected cost incurred by some decision rule $\phi(y)$ is:

$$R_j(H) = \sum_i C_{i;j} P[\phi(y) = m_i | M = m_j]$$

Therefore the overall average cost after taking prior probabilities into account is:

$$R(H) = \sum_j \pi_j R_j(H)$$

and the goal is to minimize $R(H)$.

Suppose the cost assignment is given by $C_{0;0} = C_{1;1} = 0$; $C_{1;0} = 1$, and $C_{0;1} = \gamma$. Find the decision regions for optimal minimum risk detection and investigate the behaviour when γ is very large.

PROBLEM 2. Consider an arbitrary signal set $A = \{a_j, 1 \leq j \leq M\}$. Assume that all signals are equiprobable. Let $m(A) = \frac{1}{M}\sum_j a_j$ be the average signal, and let A' be A translated by $m(A)$ so that the mean of A' is zero:

$$A' = A - m(A) = \{a_j - m(A), 1 \leq j \leq M\}.$$

Let $\mathcal{E}(A)$ and $\mathcal{E}(A')$ denote the average energies of A and A', respectively.

(a) Show that the error probability of an optimum detector for an additive channel is the same for A' as it is for A.

(b) Show that $\mathcal{E}(A') = \mathcal{E}(A) - ||m(A)||^2$. Conclude that removing the mean $m(A)$ is always a good idea.

PROBLEM 3. In this exercise we compare the power efficiency of n-cube and n-sphere signal sets for large n.

An n-cube signal set is the set of all odd-integer sequences of length n within an n-cube of side $2M$ centered on the origin. An n-sphere signal set is the set of all odd-integer sequences of length n within an n-sphere of squared radius r^2 centered on the origin.

Both n-cube and n-sphere signal sets therefore have minimum squared distance between signal points $d^2_{min} = 4$ (if they are nontrivial), and n-cube decision regions of side 2 and thus volume 2^n associated with each signal point. The point of the following exercise is to compare their average energy using the following large-signal-set approximations:

- The number of signal points is approximately equal to the volume $V(R)$ of the bounding n-cube or n-sphere region R divided by 2^n, the volume of the decision region associated with each signal point (an n-cube of side 2).

- The average energy of the signal points under an equiprobable distribution is approximately equal to the average energy $E(R)$ of the bounding n-cube or n-sphere region R under a uniform continuous distribution.

(a) Show that if R is an n-cube of side $2M$ for some integer M, then under the two above approximations the approximate number of signal points is M^n and the approximate average energy is $\frac{nM^2}{3}$. Show that the first of these two approximations is exact.

(b) For n even, if R is an n-sphere of radius r, compute the approximate number of signal points and the approximate average energy of an n-sphere signal set, using the following expressions for the volume $V_\otimes(n,r)$ and the average energy $E_\otimes(n,r)$ of an n-sphere of radius r:

$$V_\otimes(n,r) = \frac{(\pi r^2)^{\frac{n}{2}}}{(\frac{n}{2})!} \tag{3}$$

$$E_\otimes(n,r) = \frac{nr^2}{n+2} \tag{4}$$

(c) For $n = 2$, show that a large 2-sphere signal set has about 0.2 dB smaller average energy than a 2-cube signal set with the same number of signal points.

(d) For $n = 16$, show that a large 16-sphere signal set has about 1 dB smaller average energy than a 16-cube signal set with the same number of signal points. [Hint: $8! = 40320$ (46.06 dB).]

(e) Show that as $n \to \infty$ a large n-sphere signal set has a factor of $\frac{\pi e}{6}$ (1.53 dB) smaller average energy than an n-cube signal set with the same number of signal points. [Hint: Use Stirling approximation, $m! \to (\frac{m}{e})^m \sqrt{2\pi m}$ as $m \to \infty$.]

PROBLEM 4. (a) An observation y may contain information that does not help to determine which message has been transmitted. The irrelevant components may be discarded without the loss of performance. Assume the observation is of the form $y = (y_1, y_2)$. Show that for the ML receiver, if

$$P(y_2 \mid y_1, x) = P(y_2 \mid y_1)$$

then y_2 is not needed at the receiver.

(b) Consider the following channel

$$Y_1 = X + N_1$$
$$Y_2 = X + N_2$$
$$Y_3 = X + N_1 + N_2,$$

where X, N_1 and N_2 are independent random variables.

(i) Given only y_1, is y_3 irrelevant?

(ii) Given y_1 and y_2, is y_3 irrelevant?

For the rest of the problem consider the following channel

$$Y_1 = X + N_1$$
$$Y_2 = X + N_2.$$

The transmitted signal X is either -1 or $+1$. The noise random variables N_1 and N_2 are statistically independent of the transmitted signal and each other. Their density functions are

$$P_{N_1}(n) = P_{N_2}(n) = \frac{1}{2}e^{-|n|}$$

(c) Given y_1, is y_2 irrelevant?

(d) Find the optimum decision regions to minimize the probability of error for equally likely messages.

(e) A receiver decides $X = 1$ if and only if $Y_1 + Y_2 > 0$. Is this receiver optimum for equally likely messages? What is the probability of error?

(f) Find how the optimum decision regions are modified when $\Pr\{X = 1\} > \frac{1}{2}$.

Problem 5. A set of 4 orthogonal basis functions $\phi_1(t); \phi_2(t); \phi_3(t); \phi_4(t)$ is used in the following constellation. In both the first 2 dimensions and again in the second two dimensions: The constellation points are restricted such that a point E can only follow a point E and a point O can only follow a point O. The points $\{1, 1\}$, $\{-1, -1\}$ are labeled as E and $\{1, -1\}$, $\{-1, 1\}$ are labeled as O points. For instance, the 4-dimensional point $[1, 1, 1, 1]$ is permitted to occur, but the point $[1, 1, -1, 1]$ can not occur.

1. Enumerate all M points as ordered-4-tuples.
2. Find b, $\bar{b}$ (number of transmitted bits and number of transmitted bits per dimension).
3. Find E_x and $\bar{E}_x$ (energy and energy per dimension) for this constellation.
4. Find P_e for this constellation using the nearest neighbor union bound if used on an AWGN with $\sigma^2 = 0.1$.

Problem 6. Consider the following constellation to be used on an AWGN channel with variance σ^2:

$$\begin{aligned} x_0 &= (-1, -1) \\ x_1 &= (1, -1) \\ x_2 &= (-1, 1) \\ x_3 &= (1, 1) \\ x_4 &= (0, 3) \end{aligned}$$

1. Find the decision region for the ML detector.
2. Find the union bound and nearest neighbor union bound on P_e for the ML detector on this signal constellation.