ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 23 | Advanced Digital Communications
Homework 10 Solutions | December 22, 2010

PROBLEM 1. (a)

$$\begin{aligned}
\mathbb{E}[h_l(t_0)] &= \mathbb{E}[C_l \mathrm{e}^{j(2\pi\frac{v}{\lambda}\cos\theta_l t+\phi_l)}] \\
&= \mathbb{E}[C_l]\mathbb{E}[\mathrm{e}^{j(2\pi\frac{v}{\lambda}\cos\theta_l t)}]\mathbb{E}[\mathrm{e}^{j\phi_l}] \\
\mathbb{E}[\mathrm{e}^{j\phi_l}] &= \int_0^{2\pi} \mathrm{e}^{j\phi_l}\frac{1}{2\pi}d\phi_l \\
&= [-\frac{j}{2\pi}\mathrm{e}^{j\phi_l}]_0^{2\pi} = 0 \\
\mathbb{E}[h_l(t_0)] &= 0
\end{aligned}$$

(b)

$$\begin{aligned}
\mathrm{Var}[h_l(t_0)] &= \mathbb{E}[|h_l(t_0)|^2] - \underbrace{\mathbb{E}[h_l(t_0)]^2}_{=0 \text{ from (a)}} \\
&= \mathbb{E}[|C_l \mathrm{e}^{j(2\pi\frac{v}{\lambda}\cos\theta_l t+\phi_l)}|^2] \\
&= \mathbb{E}[|C_l|^2]\underbrace{\mathbb{E}[|\mathrm{e}^{j(2\pi\frac{v}{\lambda}\cos\theta_l t)}|^2]}_{=1}\underbrace{\mathbb{E}[|\mathrm{e}^{j\phi_l}|^2]}_{=1} \\
&= \frac{\sigma^2}{L} \quad \text{(second moment of } C_l)
\end{aligned}$$

(c)

$$\begin{aligned}
X_l &= \frac{\sqrt{L}}{\sigma}h_l(t_0) \\
\mathbb{E}[X_l] &= \frac{\sqrt{N}}{\sigma}\underbrace{\mathbb{E}[h_l(t_0)]}_{=0} = 0 \\
\mathrm{Var}(X_l) &= \frac{L}{\sigma^2}\underbrace{\mathrm{Var}(h_l(t_0))}_{=\frac{\sigma^2}{L}} = 1
\end{aligned}$$

Remember that $h(t) = \sum_{l=1}^{L} h_l(t)$. At time $t = t_0$:

$$\begin{aligned}
h(t_0) &= \sum_{l=1}^{L} h_l(t_0) \\
&= \sum_{l=1}^{L} X_l \frac{\sigma}{\sqrt{L}} = \sigma\frac{1}{\sqrt{L}}\sum_{l=1}^{L} X_l
\end{aligned}$$

All X_i's are i.i.d since $h_i(t_0)$'s are composed of i.i.d variables. Using the central limit theorem we get:

$$\begin{aligned}\lim_{L\to\infty} \frac{1}{\sqrt{L}} \sum_{l=1}^{L} X_l &\sim \mathcal{N}(0,1) \\ \lim_{L\to\infty} \sigma\frac{1}{\sqrt{L}} \sum_{l=1}^{L} X_l &\sim \mathcal{N}(0,\sigma^2)\end{aligned}$$

Thus for large L, $h(t)$ behaves as a zero mean Gaussian random variable of variance σ^2.

Problem 2. (a) Carrier Wavelength: $\lambda = \frac{c}{f_c} = \frac{3\times 10^8}{3\times 10^9} \frac{[\text{m/s}]}{[\text{hz}]} = 0.1 \quad [\text{m}]$

(b) Doppler Spread: $f_{max} = \frac{2\nu}{\lambda} = \frac{2\times 30}{0.1} \frac{[\text{m/s}]}{[\text{m}]} = 600 \quad [\text{hz}]$
Since the devices are moving with speed ν, the wavelength of the signal sent and received is relatively modified according to the direction of movements between sender and receiver. This time distorsion can be be represented in the frequency domain with a shift of the carrier frequency. A bound of the maximum shift is f_{max}.

(c) Coherence Time: $T_c = \frac{1}{f_{max}} = 1.7 \times 10^{-3} \quad [\text{s}]$

(d) Coherence time (T_c) is the time during which the channel remains roughly the same. The blocklength (N) must be choosen such that: $NT \ll T_c$ where T is the time needed to transmit one symbol $T = \frac{1}{W} = 1.6 \times 10^{-6}$.
Thus N is choosen lower than 1000.

(e) With $W = 12$ Mhz: $T = 8.3 \times 10^{-8}$. With this setup we obtain a bound of 20000 symbols. In terms of symbols per second there is an improvement of the rate due to a bigger bandwidth available. If the rate (in symbols per second) does not matter, we can either realize the same rate per channel use as in (d) and then allocate more symbols (N_T) for the training sequence. The channel estimation is then better than before. Alternatively, we can use the same number of timeslots N_T as in (d) for the training sequence, and then pack more data symbols into a block. This will lead to a better rate in terms of symbols per channel use.

Problem 3. 1. For one receiver, the error probability is:

$$P_e = \frac{1}{2}\left(1 - \sqrt{\frac{SNR}{1+SNR}}\right) \approx \frac{1}{2}\left(1 - \left(1 - \frac{1}{2SNR}\right)\right) = \frac{1}{4SNR}$$

Using upper bound for two receive antennas:

$$\begin{aligned} P_e &= \mathbb{E}_h Q(\frac{||h||\sqrt{\mathcal{E}_x}}{2\sigma}) \\ &\leq \mathbb{E}_h e^{-||h||^2\mathcal{E}_x/8\sigma^2} \\ &= \frac{1}{(1+\frac{\mathcal{E}_x}{8\sigma^2})^2} \\ &\approx \frac{c}{SNR^2}\end{aligned}$$

If we define "$\doteq$" notation as

$$\lim_{\mathrm{SNR}\to\infty} \frac{\log P_e(\mathrm{SNR})}{\log \mathrm{SNR}} = -d$$

written is shorthand as $P_e(\mathrm{SNR}) \doteq \mathrm{SNR}^{-d}$. Then $P_e \doteq \frac{1}{\mathrm{SNR}^2}$ above.

2.

$$P_e(\cdot|\mathcal{F}=1) \doteq \frac{1}{SNR^2}$$
$$P_e(\cdot|\mathcal{F}=0) \doteq \frac{1}{SNR}$$

3.

$$\begin{aligned} P_e &= \Pr(\mathcal{F}=1)P_e(\cdot|\mathcal{F}=1) + \Pr(\mathcal{F}=0)P_e(\cdot|\mathcal{F}=0) \\ &\doteq (1-q)\frac{1}{SNR^2} + q\frac{1}{SNR} \\ &\doteq \frac{1}{SNR} \end{aligned}$$

i.e. diversity order of 1.