ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 11 Advanced Digital Communications
Last Year's Midterm Solutions November 9, 2010

PROBLEM 1. (a) Given the observation (y_1, y_2), the maximum likelihood receiver computes for each hypothesis x

$$\text{score}(x) = p((y_1, y_2)|x) = p(y_1|x)p(y_2|y_1, x)$$

and chooses the x with the highest score. If $p(y_2|y_1, x) = p(y_2|y_1)$, then

$$\text{score}(x) = p(y_1|x)p(y_2|y_1).$$

Since the factor $p(y_2|y_1)$ is common to the score of each x, the ranking of the x's will not change if it is based on the modified score

$$\text{score}'(x) = p(y_1|x).$$

As score′ can be computed from y_1 alone, the receiver does not need y_2 to make its decision.

(b) (i). With $Y_1 = X + N_1$, $Y_2 = X + N_2$, $Y_3 = X + N_1 + N_2$ with independent X, N_1, N_2,

$$\begin{aligned}
\Pr(Y_3 \leq y_3|Y_1 = y_1, X = x) &= \Pr(X + N_1 + N_2 \leq y_3|Y_1 = y + 1, X = x) \\
&= \Pr(N_2 \leq y_3 - y_1|Y_1 = y_1, X = x) \\
&= \Pr(N_2 \leq y_3 - y_1) \qquad (*) \\
&= \Pr(Y_3 \leq y_3|Y_1 = y_1)
\end{aligned}$$

where (*) follows from the independence of N_2 from X and N_1. Thus, $p(y_3|y_1, x) = p(y_3|y_1)$ and we conclude that y_3 is irrelevant given only y_1.

(ii). Given Y_1 and Y_2, the knowledge of Y_3 would let us determine X exactly as $X = Y_1 + Y_2 - Y_3$. Such exact determination is in general not possible from Y_1 and Y_2 alone, so Y_3 is not irrelevant.

Under special circumstances the pair Y_1, Y_2 may determine X exactly, and Y_3 is irrelevant. Some examples: (1) X is a constant; (2) $N_1 = 0$ with probability 1; or perhaps more interestingly, (3) X takes only values in $\{0, 1, 2, 3, 4, 5\}$, N_1 takes only values in even integers and N_3 is always a multiple of 3, then, from Y_1 we know (X mod 2), from Y_2 we know (X mod 3), so we can find (X mod 6) and thus determine X.

(c) The conditional cumulative distribution of Y_2,

$$\Pr(Y_2 \leq y_2|Y_1 = y_1, X = x) = \Pr(N_2 \leq y_2 - x)$$

is a function that depends on the value of x. If $P(Y_2 \leq y_2|Y_1 = y_1, X = x)$ were equal to $P(Y_2 \leq y_2|Y_1 = y_1)$ this would not have been the case. So, Y_2 is not irrelevant.

(d) Observe that

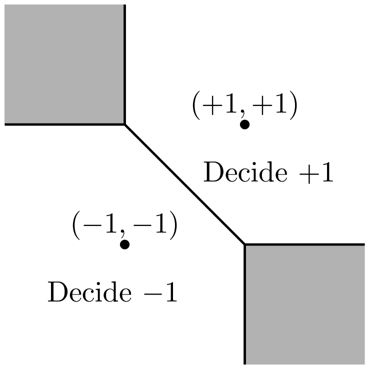
$$\log P(y_1, y_2|x) = \log P_{N_1}(y_1 - x) + \log P_{N_2}(y_2 - x) = -\big[|y_1 - x| + |y_2 - x|\big] - \log 2.$$

Thus the optimum decision rule is

$$\begin{cases} +1 & |y_1 - 1| + |y_2 - 1| < |y_1 + 1| + |y_2 + 1| \\ -1 & |y_1 - 1| + |y_2 - 1| > |y_1 + 1| + |y_2 + 1| \\ \text{either} & |y_1 - 1| + |y_2 - 1| = |y_1 + 1| + |y_2 + 1| \end{cases} = \begin{cases} +1 & g(y_1) + g(y_2) > 0 \\ -1 & g(y_1) + g(y_2) < 0 \\ \text{either} & g(y_1) + g(y_2) = 0 \end{cases}$$

with

$$\begin{aligned} g(y) &= |y+1| - |y-1| \\ &= \begin{cases} -2 & y < -1 \\ 2y & -1 \le y \le 1 \\ +2 & y > 1. \end{cases} \end{aligned}$$



The decision regions are shown in the figure with the gray zones indicating the when the decision is arbitrary.

(e) Since the rule agrees with the rule derived in part (d) it is optimum for the case of equally likely messages. By symmetry, the probability of error can be computed as $P(\text{error}) = P(\text{error}|X = -1)$, with is the same as

$$\Pr(Y_1 + Y_2 \ge 0 | X = -1) = \Pr(N_1 + N_2 \ge 2).$$

Writing the above as

$$\int p_{N_1}(n_1) P(N_2 > 2 - n_1)\, dn_1,$$

observing that

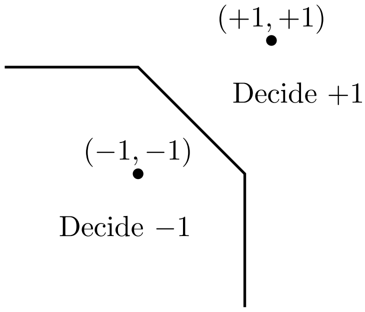
$$P(N_2 > x) = \begin{cases} \exp(-x)/2 & x \ge 0 \\ 1 - \exp(x)/2, & x < 0, \end{cases}$$

and substituting $p_{N_1}(x) = \exp(-|x|)/2$, lets us evalute the error probability as $3\exp(-2)/4$.

(f) The MAP rule is given by decision $= \arg \max_{x \in \{+1,-1\}} P(y_1, y_2|x)p(x)$,

which, with $q = \Pr(X = +1)$, simplifies to

$$\begin{cases} +1 & g(y_1) + g(y_2) > \log((1-q)/q) \\ -1 & g(y_1) + g(y_2) < \log((1-q)/q) \\ \text{either} & g(y_1) + g(y_2) = \log((1-q)/q) \end{cases}$$



With $q > 1/2$, this has the effect of eliminating the gray zone, and shrinking the decision region for $X = -1$ as shown.

Problem 2.

$$
\begin{aligned}
Q(D) &= \sum_k q_k D^k \\
&= \frac{5}{3} + \frac{5}{3}\sum_{k=1}^{\infty} 2^{-k} D^{2k} + \frac{5}{3}\sum_{k=1}^{\infty} 2^{-k} D^{-2k} + \sum_{k=1}^{\infty} 2^{-k} D^{2k-1} + \sum_{k=1}^{\infty} 2^{-k} D^{-2k+1} \\
&= \frac{5}{3} + \frac{5}{3}\frac{2^{-1}D^2}{1-2^{-1}D^2} + \frac{5}{3}\frac{2^{-1}D^{-2}}{1-2^{-1}D^{-2}} + \frac{2^{-1}D}{1-2^{-1}D^2} + \frac{2^{-1}D^{-1}}{1-2^{-1}D^{-2}} \\
&= \frac{\frac{1}{4}D + \frac{5}{4} + \frac{1}{4}D^{-1}}{(1-2^{-1}D^2)(1-2^{-1}D^{-2})} \\
&= \frac{\frac{1}{4}(a+bD^{-1})(a+bD)}{(1-2^{-1}D^2)(1-2^{-1}D^{-2})}
\end{aligned}
$$

with $a = \frac{\sqrt{7}+\sqrt{3}}{2}$ and $b = \frac{\sqrt{7}-\sqrt{3}}{2}$. The whitening filter will be

$$W(D) = \frac{1-2^{-1}D^{-2}}{a+bD^{-1}}.$$

This anti casual filter will lead to a casual channel at the output of the whitening filter, one could also consider the filter

$$W(D) = \frac{1-2^{-1}D^{2}}{a+bD^{1}}.$$

which is casual, which means its implementable, and will lead to a anti causual channel at its output. Both answers are acceptable.

Problem 3. (a) As the constellation has M points in $N = M$ dimensions is spectral efficiency is $\log_2(M)/N = \log_2(M)/M$. The energy per bit is $E_b = \mathcal{E}/\log_2 M$.

(b) The distance between signals i and j with $i \neq j$ is

$$\|a_i - a_j\|^2 = \|a_i\|^2 + \|a_j\|^2 - 2\langle a_i, a_j\rangle = 2\mathcal{E} - 2\mathcal{E}\delta_{ij} = 2\mathcal{E}$$

So the distance between any two points is $\sqrt{2\mathcal{E}}$. Thus, $d^2_{\min} = 2\mathcal{E}$, and since for any constellation point i all the other $(M-1)$ points are at this distance, each point has $(M-1)$ nearest neighbors.

(c) If signal i is sent, an error will be made if the received point is closer to some other point j. Thus,

$$\Pr(\text{Error}|i) = \Pr(\cup_{j\neq i} E_{ij}|i) \leq \sum_{j\neq i} \Pr(E_{ij}|i)$$

where E_{ij} is the event that the received point lies closer to j than i. Since

$$\Pr(E_{ij}|i) = Q(d_{ij}/(2\sigma)) = Q(\sqrt{\mathcal{E}/(2\sigma^2)})$$

we find that

$$\Pr(\text{Error}) \leq (M-1)Q\Big(\sqrt{\frac{\mathcal{E}}{2\sigma^2}}\Big)$$

(d) Writing $\mathcal{E} = (\log_2 M)E_b$, and using $Q(x) \le (2\pi x^2)^{-1/2}\exp(-x^2/2)$, we find

$$\begin{aligned}
\Pr(\text{Error}) &\le MQ(\sqrt{(E_b/2\sigma^2)\log_2 M}) \\
&\le M \exp\Bigl(-\frac{E_b}{4\sigma^2}\log_2 M\Bigr)\Big/\sqrt{\pi(E_b/\sigma^2)\log_2 M} \\
&\le \exp\Bigl(-\frac{E_b}{4\sigma^2}\log_2 M + \ln M\Bigr)\Big/\sqrt{\pi(E_b/\sigma^2)\log_2 M} \\
&\le \exp\Bigl(-\Bigl[\frac{E_b}{4\sigma^2} - \ln 2\Bigr]\log_2 M\Bigr)\Big/\sqrt{\pi(E_b/\sigma^2)\log_2 M}
\end{aligned}$$

Observe now that if $E_b/\sigma^2 > 4\ln 2$, the term in square brackets is positive and as M gets large the right hand side goes to zero exponentially fast in $\log M$.

Note that this result is shows that for reliable communication (i.e., to make Pr(Error) as small as we wish), it is not necessary to use larger and large amounts of enery per bit. As long as the amount of energy we use is larger than a fixed threshold (in our derivation $4\sigma^2 \ln 2$) the error probability can be made arbitrarily small. With a more careful derivation we can improve this threshold to $2\sigma^2 \ln 2$, in fact this turns out to be best possible.

The spectral efficiency in the limit of large M is $(\log_2 M)/M$ which approaches zero.

PROBLEM 4. (a) The bandwidth of $\hat{s}$ is the same as the bandwidth of p. The minimum bandwith pulse p that avoids intersymbol interference is the sinc pulse of bandwidth $1/(2T) = 0.5$ MHz

(b) With $p(t) = \operatorname{sinc}(t/T)$, being real

$$\begin{aligned}
s(t) &= \operatorname{Re}\Bigl\{\sum_k p(t-kT)x_k e^{j2\pi f_c t}\Bigr\} \\
&= \sum_k p(t-kT)\operatorname{Re}\Bigl\{x_k e^{j2\pi f_c t}\Bigr\} \\
&= \sum_k p(t-kT)\bigl[\operatorname{Re}\{x_k\}\cos(2\pi f_c t) - \operatorname{Im}\{x_k\}\sin(2\pi f_c t)\bigr] \\
&= \sum_k \operatorname{Re}\{x_k\}p(t-kT)\cos(2\pi f_c t) - \sum_k \operatorname{Im}\{x_k\}p(t-kT)\sin(2\pi f_c t)\bigr].
\end{aligned}$$

This would be of the form

$$\sum_k \operatorname{Re}\{x_k\}p_I(t-kT) + \sum_k \operatorname{Im}\{x_k\}p_Q(t-kT)$$

with $p_I(t) = p(t)\cos(2\pi f_c t)$ and $p_Q(t) = -p(t)\sin(2\pi f_c t)$ if $f_c T$ is an integer (so that $\cos(2\pi f_c(t-kT)) = \cos(2\pi f_c t)$ and similarly for the sin()).

(c) Observe that $\hat{s}$ occupies bandwidth $[-0.5, 0.5]$ MHz, so $\hat{s}(t)e^{j2\pi f_c t}$ occupies bandwith $[f_c - 0.5, f_c + 0.5]$. We need to ensure that this lies in $[0, 50]$ MHz, so, f_c needs to lie between 0.5 MHz and 49.5 MHz.

(d) Observe that $(h * s)(t) = s(t)$ and it occupies frequencies f with $|f| \in [f_c - 0.5, f_c + 0.5]$ MHz. The total bandwidth occupied by the signal is 2 MHz (taking into account both positive and negative f's). Thus the total noise power in these bands is 2 MHz $\times$ 0.5×10^{-11} W/Hz $= 10^{-5}$ W, making the signal to noise ratio 10^5. We thus have

$$d^2(M-1)/6 = 10^5\sigma^2,$$

equivalently

$$[d/(2\sigma)]^2 = 1.5 \times 10^5/(M-1) \qquad (*)$$

Using the hint, to upper bound the probability of error by 10^{-7} it suffices to ensure $4Q(d/(2\sigma)) \leq 10^{-7}$, which requires a $d/(2\sigma)$ slightly larger than 5.45. Plugging this in to $(*)$ gives $M = 5051$ as the maximum possible size of a M-QAM constellation. Requiring M to be the square of an even number makes $M = 70^2 = 4900$ as the size of largest M-QAM constellation that satisfies the error probability requirement. Since each constellation points carries $\log_2(M) = 12.25$ bits of information, the data rate is $R = 12.25$ Mbit/s.

(e) The bandwidth occupied by $\hat{s}$ is $[-1/(2T), 1/(2T)]$ is an interval of length $1/T$. If the signal s is to fit in $[-50, 50]$ MHz we then require the bandwidth of $\hat{s}$ to occupy at most an interval of size 50 MHz, which constraints $T \geq 20$nsec. Denoting $B = 1/T$, and mesuring it in MHz, the computation just as in part (d) above gives the signal to noise ratio as $10^5/B$. Also just as above, we find that we need

$$\frac{1.5 \times 10^5}{B(M-1)} \geq (5.45)^2$$

which limits M to $1 + 5050/B$ (ignoring the square of an even number constraint). The data rate is thus

$$R = (1/T)\log_2 M = B \log_2(1 + 5050/B),$$

which is an increasing function of B, and thus is attained at the largest possible value of B which is 50. Thus corresponding $M = 102$, $T = 20$ns, and $R = 333$ Mbits/s. (If we set $M = 100$ to make it equal an even square, we get $R = 332$ Mbits/s.)