ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 10 Advanced Digital Communications
Last Year's Midterm Exam November 9, 2010

PROBLEM 1. (30 points)

(a) (5 pts) An observation y may contain information that does not help to determine which message has been transmitted. The irrelevant components may be discarded without the loss of performance. Assume the observation is of the form $y = (y_1, y_2)$. Show that for the ML receiver, if

$$P(y_2 \mid y_1, x) = P(y_2 \mid y_1)$$

then y_2 is not needed at the receiver.

(b) (5 pts) Consider the following channel

$$\begin{aligned} Y_1 &= X + N_1 \\ Y_2 &= X + N_2 \\ Y_3 &= X + N_1 + N_2, \end{aligned}$$

where X, N_1 and N_2 are independent random variables.

(i) Given only y_1, is y_3 irrelevant?

(ii) Given y_1 and y_2, is y_3 irrelevant?

For the rest of the problem consider the following channel

$$\begin{aligned} Y_1 &= X + N_1 \\ Y_2 &= X + N_2. \end{aligned}$$

The transmitted signal X is either -1 or $+1$. The noise random variables N_1 and N_2 are statistically independent of the transmitted signal and each other. Their density functions are

$$P_{N_1}(n) = P_{N_2}(n) = \frac{1}{2} e^{-|n|}$$

(c) (5 pts) Given y_1, is y_2 irrelevant?

(d) (5 pts) Find the optimum decision regions to minimize the probability of error for equally likely messages.

(e) (5 pts) A receiver decides $X = 1$ if and only if $Y_1 + Y_2 > 0$. Is this receiver optimum for equally likely messages? What is the probability of error?

(f) (5 pts) Find how the optimum decision regions are modified when $\Pr\{X = 1\} > \frac{1}{2}$.

PROBLEM 2. (25 points) Consider transmission over ISI channel and the channel after match filtering is

$$Y(D) = \|p\| Q(D) X(D) + Z(D).$$

Let

$$q_k = \begin{cases} 2^{\frac{-|k|-1}{2}} & k \text{ odd;} \\ \frac{5}{3} 2^{\frac{-|k|}{2}} & k \text{ even;} \end{cases}$$

and $S_z(D) = N_0 Q(D)$ and $Q(D)$ is the D-transform of q_k. Find the whitening filter to whiten the noise so that the resulting channel after whitening is causal.

PROBLEM 3. (25 points) An orthogonal signal constellation is a set $A = \{a_j, 1 \leq j \leq M\}$ of M orthogonal vectors in $\mathcal{R}^M$ with equal energy $\mathcal{E}$; i.e., $\langle a_j, a_i \rangle = \mathcal{E}\delta_{ji}$.

(a) (5pts) Compute the nominal spectral efficiency ρ of A in bits per dimension. Compute the average energy E_b per information bit.

Hint. In a N-dimensional M-point constellation, we define,

- The bit rate (nominal spectral efficiency) $\rho = \frac{1}{N}\log_2 M$ b/D
- The average energy per dimension $E_s = \frac{1}{N}\mathcal{E}$, or the average energy per bit $E_b = \frac{\mathcal{E}}{\log_2 M} = \frac{E_s}{\rho}$.

(b) (5 pts) Compute the minimum squared distance $d^2_{\min}(A)$. Show that every signal has $M-1$ nearest neighbors.

(c) (5 pts) Let the noise variance be σ^2 per dimension. Show that the probability of error of an optimum detector is bounded by the union bound error

$$\Pr(E) \leq (M-1)Q\Big(\sqrt{\frac{2\mathcal{E}}{\sigma^2}}\Big).$$

(d) (10 pts) Let $M \to \infty$ with E_b held constant. Using an asymptotically accurate upper bound for Q function, show that $\Pr(E) \to 0$ provided that $\frac{E_b}{\sigma^2} > 4\ln 2$. What is the nominal spectral efficiency ρ in the limit?

Hint.

$$\begin{aligned} Q(x) &\leq \frac{1}{\sqrt{2\pi x^2}}e^{-\frac{x^2}{2}} \\ Q(x) &\geq \big(1-\frac{1}{x^2}\big)\frac{1}{\sqrt{2\pi x^2}}e^{-\frac{x^2}{2}}. \end{aligned}$$

Problem 4. (30 points) Consider a M-QAM constellation $A \subset \mathbb{C}$ with $M = (2m)^2$ points,

$$A = \big\{a\colon \operatorname{Re}\{a\} \in \{\pm\tfrac{1}{2}d, \pm\tfrac{3}{2}d, \dots, \pm\tfrac{2m-1}{2}d\}, \\ \operatorname{Im}\{a\} \in \{\pm\tfrac{1}{2}d, \pm\tfrac{3}{2}d, \dots, \pm\tfrac{2m-1}{2}d\}\big\}.$$

The average energy of this constellation is given by $d^2(M-1)/6$.

The constellation A is used to generate a complex baseband signal

$$\hat{s}(t) = \sum_k x_k p(t - kT), \quad x_k \in A,$$

with $T = 10^{-6}$sec and $p(t)$ a unit energy pulse.

(a) (5 pts) What is the minimum possible bandwith of the signal $\hat{s}$ if inter-symbol interference is to be avoided?

(b) (5 pts) Suppose that $\hat{s}$ is as in part (a). Let $f_c > 0$ and set $s(t) = \sqrt{2}\operatorname{Re}\{\hat{s}(t)e^{j2\pi f_c t}\}$ as the signal transmitted signal over a channel. If we express $s(t)$ as

$$s(t) = \sum_k \operatorname{Re}\{x_k\} p_I(t - kT) + \sum_k \operatorname{Im}\{x_k\} p_Q(t - kT)$$

find p_I and p_Q.

(c) (5 pts) Suppose the channel has an impulse response $h(t)$ with

$$H(f) = \begin{cases} 1 & |f| \le 50\text{MHz} \\ 0 & \text{else.} \end{cases}$$

What are the minimum and maximum possible values of f_c if s is not to be distorted by the channel?

(d) (5 pts) Suppose f_c is chosen as in part (c), M and d are chosen so that $s(t)$ has power 1Watt, and that the received signal is $y(t) = (h * s)(t) + z(t)$ where z is white Gaussian noise with spectral density 5×10^{-11}Watts/Hz. How large can M be so that the probability of error is kept below 10^{-7}? What is the corresponding data rate R in bits/sec? *Hint.* The error probability for a M-QAM constellation with minimum distance d observed with additive Gaussian of variance σ^2 is upper bounded by $4Q\big(\frac{d}{2\sigma}\big)$.

(e) (10 pts) Suppose the channel is as in part (d). Suppose we are now free to modify our communication system by adjusting T and f_c, but keeping to M-QAM modulation, intersymbol interference free $\hat{s}$, and s undistorted by the channel filter h and of power 1Watt. What is the largest data rate that can be achieved while keeping the probability of error less than 10^{-7}?