ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 11 Signal Processing for Communications
Homework 7 April 12, 2010

Problem 1 (Excercise 6.4 of textbook). (I) Consider a causal LTI system with the following transfer function:

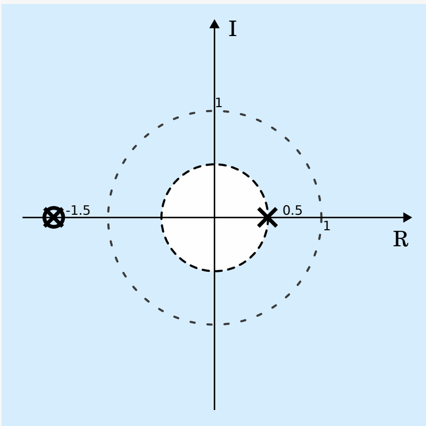
$$H(z) = \frac{3 + 4.5z^{-1}}{1 + 1.5z^{-1}} - \frac{2}{1 - 0.5z^{-1}}$$

Sketch the pole-zero plot of the transfer function and specify its region of convergence. Is the system stable?

We can rewrite the transfer function as :

$$H(z) = 3 - \frac{2}{1 - 0.5z^{-1}}$$

where we can immediately see that there is only one pole at $z = 0.5$. Mathematically speaking the pole and the zero cancel each other out, but it is better to mention them in zero-pole diagram. Since we want our system to be causal, the ROC should extend from the outernmost pole to infinity.



Since the unit circle is inside the ROC, we know that the system is stable.

(II) Consider the transfer function of an anticausal LTI system

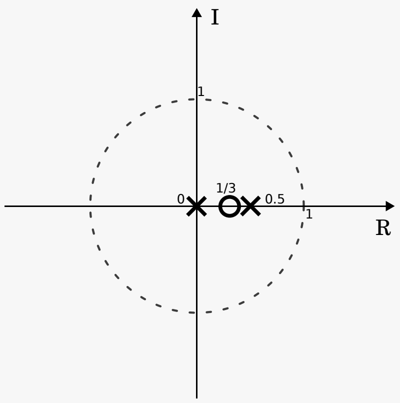
$$H(z) = (1 - z^{-1}) - \frac{1}{1 - 0.5z^{-1}}$$

Sketch the pole-zero plot of the transfer function and specify its region of convergence. Is the system stable?

First, let's rewrite $H(z)$:

$$H(z) = \frac{(1 - 0.5z^{-1})(1 - z^{-1}) - 1}{1 - 0.5z^{-1}} = -1.5z^{-1}\frac{1 - \frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

This transfer function has a pole at $z = 0$, a pole at $z = \frac{1}{2}$ and a zero at $z = \frac{1}{3}$.



For an anticausal system, the ROC has to include zero, but since there is a pole, the ROC is empty. This means that there is no anticausal system which has this transfer function. We cannot answer the stability question.

Problem 2 (Excercise 7.2 of textbook). Assume $\mathcal{G}$ is stable, causal IIR filter with impulse response $g[n]$ and transfer function $G(z)$. Which of the following statements is/are true for any choice of $G(z)$?

(a) The inverse filter, $1/G(z)$, is stable.

False.

When you take the inverse of $G(z)$, the poles become zeros and vice versa. The ROC of $1/G(z)$ is thus shaped by the zeros of $G(z)$, but we don't know anything about them. It might be possible that the unit circle is in $1/G(z)$'s ROC, but we can't tell.

(b) The inverse filter is FIR.

False.

From the book (page 154) we know that FIR filters cannot contain poles in their transfer functions. But since we cannot exclude that $G(z)$ contains zeros, we cannot be sure that the inverse contains no poles. It could therefore be IIR.

(c) The DTFT of $g[n]$ exists.

True.

Recall that the DTFT exists only if $g[n]$ is absolutely summable, $\sum_{n=-\infty}^{\infty} |g[n]| < \infty$. Since the system is stable the Z-transform exists on the unit circle, which means the series $\sum x[n]e^{j\theta n}$ converges for any value of θ, implying that $\sum x[n]^2$ is finite.

(d) The cascade $G(z)G(z)$ is stable.

True.

The cascade (multiplication) is stable because it contains the same poles and zeros than $G(z)$. The ROC cannot change, and if $G(z)$ is stable, then so is the cascade.

Problem 3 (Excercise 7.3 of textbook). Consider $G(z)$, the transfer function of a causal stable LTI system. Which of the following statements is/are true for any such $G(z)$?

(a) The zeros of $G(z)$ are inside the unit circle.

False.

We cannot say anything about the existence or positions of the zeros from the description of $G(z)$

(b) The ROC of $G(z)$ includes the curve $|z| = .5$.

False.

There might be a pole between with $0.5 < |z| < 1$, which would force the ROC to be from $|z|$ to ∞, in which case $G(z)$ would satisfy the description, but not point (b).

(c) The system $H(z) = (1 - 3z^{-1})G(z)$ is stable.

True.

$1 - 3z^{-1}$ does not create a new pole (only a zero), and does not influence stability. Because $G(z)$ is stable, $H(z)$ is too.

(d) The system is an IIR filter.

False.

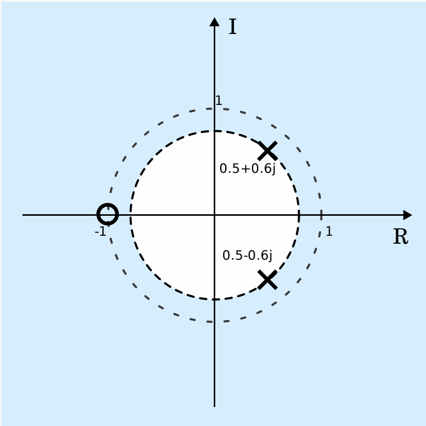
There exists filters $G(z)$ which do not contain poles, and therefore are FIR. (Take for instance $G(z) = 1$, a filter which does nothing.)

Problem 4 (Excercise 7.9 of textbook). Consider a causal IIR filter with the following transfer function:

$$H(z) = \frac{1 + z^-1}{1 - 1.6\cos(2\pi/7)z^-1 + .64z^{-2}}$$

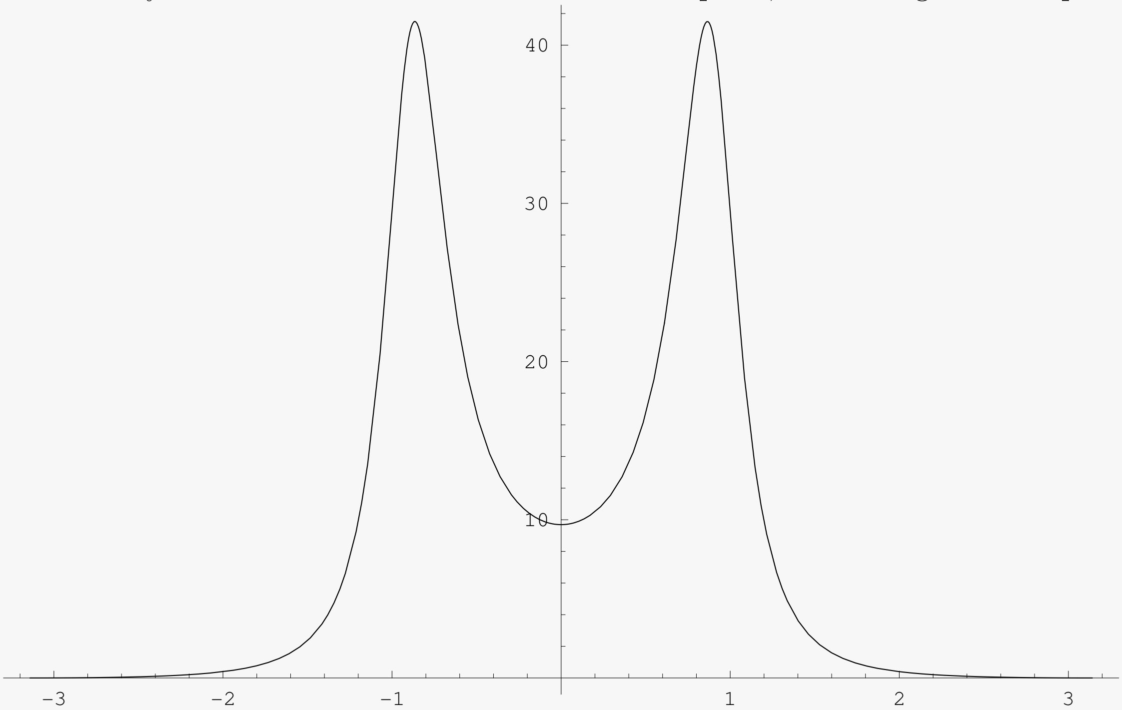
(a) Sketch the pole-zero plot of the filter and the ROC of its transfer function.

Using your favourite equation solver (try with `Solve[1 - 16/10 Cos[2 pi/7] z ^ (-1) + z ^ (-2)*64/100, z]` on wolfram alpha), you'll find the two symmetrically located poles $z \approx 0.5 \pm 0.6i$. There also is a zero at $z = -1$.



(b) Sketch the magnitude of its frequency response.

The following plot shows $|H(e^{jt})^2|$ with t between $-\pi$ and π. We can clearly see the effect of the two poles, creating the spikes.



(c) Draw at least two different block diagrams which implement the filter.

We begin by rewriting our transfer function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{(1 - pz^{-1})(1 - p^{\star}z^{-1})}$$

with $p \approx 0.5 + 0.6j$.

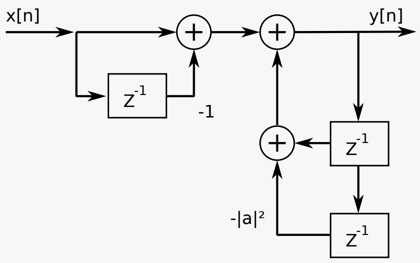
$$H(z) = \frac{1 - z^{-1}}{(1 - 2\Re(p)z^{-1} + |p|^2 z^{-2})}$$

$$(1 - z^{-1})X(z) = (1 - 2\Re(p)z^{-1} + |p|^2 z^{-2})Y(z)$$

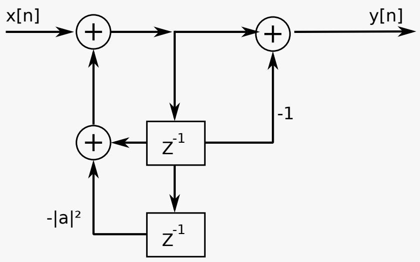
$$x[n] - x[n-1] = y[n] - 2 \cdot 0.5y[n-1] + |p|^2 y[n-2]$$

$$y[n] = x[n] - x[n-1] + y[n-1] - |p|^2 y[n-2]$$

We can now draw a block diagram :



We are allowed to exchange the left part with the right one (Book, chapter 7.4.2), which creates the following block diagram:



(d) Compute the first five values of the signal $y[n] = h[n] * x[n]$, where $x[n] = \delta[n] + 2\delta[n-1]$. Assume zero initial conditions.

It is simple to calculate the values by hand using the equation from point (c).

```
y[0] = 1.0
y[1] = 2.0
y[2] = -0.61
y[3] = -1.83
y[4] = -1.4579
y[5] = -0.3416
```

Problem 5 (Excercise 7.11 of textbook). In data communication systems over phone lines (such as voiceband modems), one of the major problems is represented by echos. Impedance mismatches along the analog line create delayed and attenuated replicas of the transmitted signal. These replicas are added to the original signal and represent one type of distortion.

Assume a simple situation where a single echo is created; the transmitted signal is $x[n]$ and, because of the echo, the received signal is

$$y[n] = x[n] - \alpha x[n-D]$$

where α is the attenuation factor (with $0 < \alpha < 1$) and D is the echo delay (assume D is an integer).

(a) Write the transfer function $H(z)$ of the echo system, i.e. the system which produces $y[n]$ from $x[n]$.

Using the fact that the Z-transform is a linear transform, and using the formula for the time-shift (if $X(z)$ is the Z-transform of $x[n]$, then z^{-D} is the transform of $x[n-D]$), we can write :

$$Y(z) = X(z) - \alpha z^{-D} X(z)$$

$$\begin{aligned} Y(z) &= H(z)X(z) \\ &= (1 - \alpha z^{-D})X(z) \end{aligned}$$

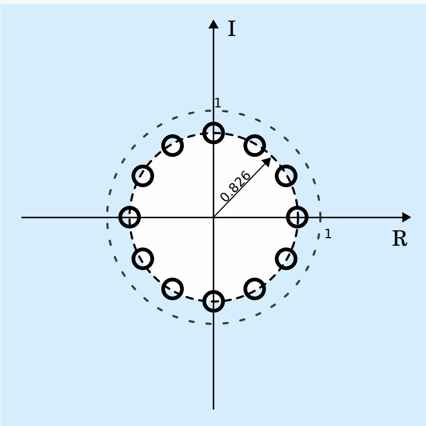
And we can read $H(z) = 1 - \alpha z^{-D}$.

(b) Sketch the pole-zero plot for $H(z)$ for $\alpha = 0.1$ and $D = 12$ (for our purposes, assume $(0.826)^{12} = 0.1$).

$H(z)$ has no poles, only zeros. To find them, we proceed as follows :

$$1 - 0.1z^{-12} = 0 \Rightarrow 10 = z^{-12} \Rightarrow 0.1 = z^{12}$$

This equation has 12 complex roots for z, which are $z = 0.826e^{j\frac{2\pi}{12}k}$ where $k = 0, 1, \ldots, 11$.



(c) Sketch the squared magnitude response $|H(e^{j\omega})|^2$.

Now assume we have a good estimate of α and D; we want to design a causal echo cancellation filter, i.e. a filter with causal impulse response $g[n]$ such that $y[n] * g[n] = x[n]$.

(d) Write the expression for $G(z)$.

We want that

$$Y(z)G(z) = X(z)$$

$$X(z)H(z)G(z) = X(z)$$

And we can cancel out the $X(z)$, resulting in

$$H(z)G(z) = 1$$

Therefore, $G(z)$ has to be the inverse of $H(z)$, and we get

$$G(z) = \frac{1}{1 - \alpha z^{-D}}$$

(e) Sketch its pole-zero plot and its ROC for the same values of α and D as before.

The plot is the same as for $H(z)$, except that there are poles instead of zeros.

(f) What is the practical difficulty in implementing this echo cancellation system?

It is very difficult to make sure that the poles precisely cancel the zeros out, even a slight difference, which could for instance be caused by numerical or quantization errors, can have a big impact on the filter. Furthermore, it is impossible to have a channel estimate which is perfectly correct.