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Handout 11 Signal Processing for Communications
Homework 5 Solution March 29, 2010

Problem 1.

$$x[n] = u[n] \to y[n] = h[n] * x[n] = \sum_k h[k]x[n-k] = \sum_k a^{-k}u[-k]u[n-k]$$

If we consider negative and positive n's, we will have:
for $n < 0$,

$$y[n] = \sum_{k=-\infty}^{n} a^{-k} = \sum_{k=-n}^{\infty} a^k = \frac{a^{-n}}{1-a}$$

and for $n \geq 0$

$$y[n] = \sum_{k=-\infty}^{0} a^{-k} = \sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$$

so, we can write the answer as follow:

$$y[n] = \frac{1}{1-a}(u[n] + a^{-n}u[-n-1])$$

Problem 2. In order to find the correct answer for this question, let's take a look at how convolution works. For computing the output at point n i.e. $y[n] = \sum_k x[k]h[n-k]$ you just need to consider signals $x[k]$ and $h[k]$, flip one of them and shift it to the right by n, and then sum up the multiplication of the corresponding coefficients. If the supports of $x[k]$ and $h[k]$ are N and M respectively (for simplicity, assume that they are from 0 up to $N-1$ and $M-1$ respectively), the maximum value of n for which $y[n]$ may not be equal to zero would be $M+N-2$. In fact the support length of y would be $M+N-1$ which is equal to the length of $[0, ..., M+N-2]$.

Problem 3. (i)

$$causality \to h[n] = 0 \;\; for \;\; n < 0$$

(ii)

$$H(e^{j\omega}) = H^*(e^{-j\omega}) \to h[n] = h^*[n] \to h[n] \;\; is \;\; real$$

(iii)

$$\begin{aligned} \text{DTFT}\{h[n+1]\} &= \text{DTFT}^*\{h[n+1]\} \to H(e^{j\omega})e^{j\omega} = H^*(e^{j\omega})e^{-j\omega} \\ \to H^*(e^{j\omega}) &= H(e^{j\omega})e^{2j\omega} \to h^*[-n] = h[n+2] \\ \to h^*[-n] &= h[-n] = h[n+2] \end{aligned}$$

Where the last line is concluded from (ii). Now, we can say surely that $h[n]$ is equal to zero for $n \geq 3$ and $n < 0$ which means the impulse response has finite duration.

(iv)

$$\frac{1}{2}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega \rightarrow h[0] = 2$$

Considering the result of (iii), we can say that $h[2] = h[0] = 2$ because $h[-n] = h[n+2]$ (just put $n = 0$).

(v)

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=0}^{2} h[n]e^{-j\omega n} \\ &= 2 + h[1]e^{-j\omega} + 2e^{-2j\omega} \\ \rightarrow H(e^{j\pi}) &= 2 + h[1]e^{-j\pi} + 2e^{-j2\pi} = 4 - h[1] = 0 \\ \rightarrow h[1] &= 4 \end{aligned}$$

So, $h[n]$ is completely determined and it is equal to

$$h[n] = 2\delta[n] + 4\delta[n-1] + 2\delta[n-2]$$

PROBLEM 4. (a) Using DTFT-properties in the book:

$$\text{DTFT}^{-1}\ Y^*(e^{j\omega}) = y^*[-n]$$

$$\text{DTFT}^{-1}\ X(e^{j\omega})Y^*(e^{j\omega}) =\ x[n] * y^*[-n]$$

(b) Assume that $h[n] = x[n] * y^*[-n]$

$$h[n] = \sum_{k=-\infty}^{+\infty} x[n-k]y^*[-k] \rightarrow h[0] = \sum_{k=-\infty}^{+\infty} x[-k]y^*[-k] = \sum_{k=-\infty}^{+\infty} x[k]y^*[k]$$

On the other hand, we have:

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} H(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}X(e^{j\omega})Y^*(e^{j\omega})e^{j\omega n}d\omega \rightarrow h[0] = \frac{1}{2\pi}X(e^{j\omega})Y^*(e^{j\omega})d\omega$$

So, we will have this equality:

$$\sum_{k=-\infty}^{+\infty} x[k]y^*[k] = \frac{1}{2\pi}X(e^{j\omega})Y^*(e^{j\omega})d\omega$$

(c) We define $sinc(x) = \frac{sin(x)}{x}$, So, we have:

$$x[n] = \frac{\sin\frac{\pi n}{4}}{2\pi n} = \frac{1}{8}sinc(\frac{\pi n}{4})$$

$$y^*[n] = y[n] = \frac{\sin\frac{\pi n}{6}}{5\pi n} = \frac{1}{30}sinc(\frac{\pi n}{6})$$

hint

$$\text{DTFT}\ \frac{1}{N} sinc(\frac{\pi n}{N}) = 1 \quad for \quad |\omega| < \frac{1}{N}$$

$$\rightarrow X(e^{j\omega}) = \frac{1}{2} \quad for \quad |\omega| < \frac{\pi}{4}$$

$$\rightarrow Y(e^{j\omega}) = \frac{1}{5} \quad for \quad |\omega| < \frac{\pi}{6}$$

Using the result of previous part:

$$\sum_{n=-\infty}^{+\infty} \frac{\sin \frac{\pi n}{4}}{2\pi n} \frac{\sin \frac{\pi n}{6}}{5\pi n} = \frac{1}{2\pi} \int_{-\frac{\pi}{6}}^{+\frac{\pi}{6}} \frac{1}{2} . \frac{1}{5}\ d\omega = \frac{1}{60}$$

PROBLEM 5. (i)

$$\begin{aligned} X(z) &= \sum_n a^n u[n] z^{-n} + \sum_n b^n u[n] z^{-n} + \sum_n c^n u[-n-1] z^{-n} \\ &= \sum_{n=0}^{+\infty} (az^{-1})^n + \sum_{n=0}^{+\infty} (bz^{-1})^n + \sum_{n=-\infty}^{-1} (cz^{-1})^n \\ &= \frac{1}{1-az^{-1}} + \frac{1}{1-bz^{-1}} + \sum_{n=1}^{+\infty} (c^{-1}z)^n \\ &= \frac{1}{1-az^{-1}} + \frac{1}{1-bz^{-1}} + \frac{c^{-1}z}{1-c^{-1}z} \\ &= \frac{1}{1-az^{-1}} + \frac{1}{1-bz^{-1}} - \frac{1}{1-cz^{-1}} \end{aligned}$$

$$ROC = \bigcap [\ |az^{-1}| < 1,\ |bz^{-1}| < 1,\ |c^{-1}z| < 1\ ] = \bigcap\ [|a| < |z|,\ |b| < |z|,\ |z| < |c|\ ]$$

where $\bigcap$ means intersection. Since $|a| < |b| < |c| \rightarrow ROC = |b| < |z| < |c|$ which is a ring between $|b|$ and $|c|$.

There are 3 poles at $p_1 = a, p_2 = b, p_3 = c$, and one triple-zero at $z = 0$

(ii)

$$X(z) = \sum_n \frac{(-1)^n}{n!} z^{-n} u[n] = \sum_{n=0}^{+\infty} \frac{(-z)^n}{n!} = e^{-z}$$

$$ROC = z - plane$$

There is just one zero at $+\infty$, because $\lim_{z\rightarrow+\infty} X(z) = 0$

(iii)

$$X(z) = \sum_n \frac{-1}{n} z^{-n} u[n-1] = -\sum_{n=1}^{+\infty} \frac{(z^{-1})^n}{n} = \ln(1-z^{-1})$$

$$ROC = |z^{-1}| < 1 = |z| > 1$$

There is just one zero at $+\infty$, because $\lim_{z\rightarrow+\infty} X(z) = 0$

PROBLEM 6. (i)

$$y[n] + \frac{1}{15}y[n-1] - \frac{2}{5}y[n-2] = x[n]$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1 + \frac{1}{15}z^{-1} - \frac{2}{5}z^{-2}}$$

$$H(e^{j\omega}) = \frac{1}{(1 + \frac{2}{3}z^{-1})(1 - \frac{9}{15}z^{-1})} = \frac{\frac{10}{19}}{1 + \frac{2}{3}z^{-1}} + \frac{\frac{9}{19}}{1 - \frac{9}{15}z^{-1}}$$

(ii)

This system has 2 poles in $p_1 = -\frac{2}{3}$, $p_2 = \frac{9}{15}$ and one double-zero at $z = 0$. The system is causal so ROC must be in form of $|z| > R$ where R is the radius correspond to the largest pole.

$$ROC = \bigcap [\ |z| > \frac{2}{3},\ |z| > \frac{9}{15}\ ] = |z| > \frac{2}{3}.$$

This system is stable, because ROC contains the unit circle.

(iii)

Yes, Since $H(e^{j\omega}) = H(z)|_{z=1}$ and the system is stable (which means ROC contains the unit circle) we can conclude that the Fourier transform will converge.

(iv)

$$h[n] = (\frac{10}{19}(-\frac{2}{3})^n + \frac{9}{19}(\frac{9}{15})^n)u[n]$$