ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 28 — Signal Processing for Communications

Homework 11 Solution — April 17, 2010

Problem 1. When a random process $X[n]$ passes through the filter $h[n]$ to obtain a random process $Y[n]$, we will have:

$$r_y[n] = (h[n] * h[-n]) * r_x[n], \quad r_{xy}[n] = h[n] * r_x[n]$$

and in Fourier domain:

$$P_y(e^{j\omega}) = |H(e^{j\omega})|^2 P_x(e^{j\omega}), \quad P_{xy}(e^{j\omega}) = H(e^{j\omega})P_x(e^{j\omega})$$

In this problem we have $h[n] = \delta[n] + \delta[n-1]$, so:

$$h[n] * h[-n] = \delta[n+1] + 2\delta[n] + \delta[n-1]$$

and since $r_x[n] = \sigma^2\delta[n]$, we can conclude:

$$r_y[n] = \sigma^2\delta[n+1] + 2\sigma^2\delta[n] + \sigma^2\delta[n-1]$$

Problem 2. (a)

$$Y[n] = X[n] + \beta X[n-1] \rightarrow Y[n] = X[n] * h[n], \quad where \quad h[n] = \delta[n] + \beta\delta[n-1]$$

$$H(e^{j\omega}) = 1 + \beta e^{-j\omega} \rightarrow |H(e^{j\omega})|^2 = 1 + \beta^2 + 2\beta\cos(\omega)$$

We know that:

$$\begin{aligned}
\sum_{k=-\infty}^{+\infty} a^{|k|}e^{-j\omega k} &= 1 + \sum_{k=1}^{+\infty}(ae^{-j\omega})^k + \sum_{k=-\infty}^{-1}(a^{-1}e^{-j\omega})^k \\
&= 1 + \sum_{k=1}^{+\infty}(ae^{-j\omega})^k + (ae^{j\omega})^k \\
&= 1 + \frac{ae^{-j\omega}}{1-ae^{-j\omega}} + \frac{ae^{j\omega}}{1-ae^{j\omega}} \\
&= \frac{1-a^2}{1+a^2-ae^{j\omega}-ae^{-j\omega}} \\
&= \frac{1-a^2}{1+a^2-2a\cos(\omega)}
\end{aligned}$$

since $R_x[k] = \delta^2 a^{|k|}$, we have:

$$P_x(e^{j\omega}) = \delta^2\frac{1-a^2}{1+a^2-2a\cos(\omega)} \rightarrow P_y(e^{j\omega}) = \delta^2\frac{1-a^2}{1+a^2-2a\cos(\omega)}(1+\beta^2+2\beta\cos(\omega))$$

(b) If $\beta = -a$, then $P_y(e^{j\omega}) = \delta^2(1-a^2)$ which is a constant value, so $Y[n]$ would be a white noise random process. In fact, in this case noise power is distributed uniformly on all frequencies.

PROBLEM 3. I call signal power and noise power S and N respectively. So, we will have:

$$S = \int_{-1}^{2} x^2 \ dx = \frac{x^3}{3} \ |_{-1}^{2} = \frac{8}{3} + \frac{1}{3} = 3$$

$$N = \int_{-1}^{0} (x - (-1))^2 \ dx + \int_{0}^{2} (x-1)^2 \ dx = \frac{1}{3} + \frac{2}{3} = 1$$

so signal to noise ratio would be equal to $\frac{S}{N} = 3$

PROBLEM 4. (a) $f_X(x)$ is a distribution function, so the integration of this function over $[0,1]$ should be equal to 1. so we have:

$$\int_{0}^{1} (\frac{1}{2} + bx) \ dx = \frac{1}{2}x + b\frac{x^2}{2} \ |_{0}^{1} = \frac{1}{2} + b\frac{1}{2} = 1 \to b = 1$$

(b) I denote the first r bits of binary expansion of $x \in [0,1]$ by a r-tuple vector $\hat{x}$. So, $\hat{x} \in \{0,1\}^r$.

I divide $[0,1]$ to 2^r subinterval $I_i = [(i-1)2^{-r}, i2^{-r})$ and $i = 1, ..., 2^r$.

$$Pr(\hat{X} = \hat{x}) = Pr(X \in I_i) = \int_{(i-1)2^{-r}}^{i2^{-r}} (\frac{1}{2} + x) \ dx = \frac{1}{2}(x + x^2) \ |_{(i-1)2^{-r}}^{i2^{-r}}$$

$$= \frac{1}{2}(i2^{-r} + i^2 2^{-2r} - (i-1)2^{-r} - (i-1)^2 2^{-2r}) = \frac{1}{2}(2^{-r} + (2i-1)2^{-2r}) = (2^{-r-1} - 2^{-2r-1}) + (2^{-2r})i$$

In fact, since $f_X(x)$ is increasing function with respect to x, $Pr(X \in I_i)$ is also an increasing function with respect to i.