ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 25 | Signal Processing for Communications
Solution to homework 10 | May 3, 2010

PROBLEM 1. Prove the following two identities:

(1) Downsampling by 2 followed by filtering by $H(z)$ is equivalent to filtering by $H(z^2)$ followed by downsampling by 2.

We can see that after downsampling, we get $X_D(z) = X(z^{1/2})$, which we filter with $H(z)$, and we get as output $H(z)X(z^{1/2})$.

If we first filter by $H(z^2)$, we get $X_{filtered} = X(z)H(z^2)$, which gets downsampled, and we get as output $X(z^{1/2})H(z)$.

The same can be shown graphically by looking at the spectra: remember that the DTFT of a signal can be read off the Z-transform by looking at the unit circle. If you consider the effect of squaring z (line in $X(z^2)$) on the unit circle, you will notice that the square squeezes the DTFT on the frequency axis.

(2) Filtering by $H(z)$ followed by upsampling by 2 is equivalent to upsampling by 2 followed by filtering by $H(z^2)$.

Following a similar method as for the first part, it is easy to see that the output in both cases is $X(z^2)H(z^2)$.

PROBLEM 2. Consider the following block diagram:

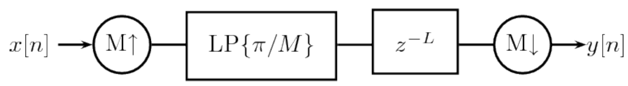


Figure 1: Problem 2

and show that this system implements a fractional delay (i.e. show that the transfer function of the system is that of a pure delay, where the delay is not necessarily an integer).

First, consider the first two blocks (upsampler and low-pass filter), after the upsampler, the spectrum contains M replicas of the original spectrum. These replicas come from the 2π periodicity of $X(e^{j\omega})$. The low-pass filter, cuts off all replicas except the one between 0 and $\frac{\pi}{M}$. It is clear that with $L = 0$ (i.e. without any delay), the downsampler restores exactly the input signal $x[n]$.

The delay has no influence on the magnitude of the DTFT, it only influences the phase. Even with non-zero L, the output spectrum magnitude is still identical to the input signal spectrum magnitude. Using the insight from problem 1. , we can swap the delay and the downsampler, and we get $X_{out}(z) = z^{-L/M}X(z)$.

To see a practical use of this structure, consider now a data transmission system over an analog channel. The transmitter builds a discrete-time signal $s[n]$; this is converted to an analog signal $s_c(t)$ via an interpolator with period T_s, and finally $s_c(t)$ is transmitted over the channel. The signal takes a finite amount of time to travel all the way to the receiver; say that the transmission time over the channel is t_0 seconds: the received signal $\hat{s}_c(t)$ is therefore just a delayed version of the transmitted signal,

$$\hat{s}_c(t) = s_c(t - t_0)$$

At the receiver, $\hat{s}_c(t)$ is sampled with a sampler with period T_s so that no aliasing occurs to obtain $\hat{s}[n]$.

1. Write out the Fourier Transform of $\hat{s}_c(t)$ as a function of $S_c(j\Omega)$.
2. Write out the DTFT of the received signal sampled with rate T_s, $\hat{s}_c[n]$.
3. Now we want to use the above multirate structure to compensate for the transmission delay. Assume $t_0 = 4.6T_s$; determine the values for M and L in the above block diagram so that $\hat{s}[n] = s[n - D]$, where $D \in \mathcal{N}$ has the smallest possible value (assume an ideal lowpass filter in the multirate structure).

PROBLEM 3. Consider a discrete-time signal $x[n]$ with the following spectrum:

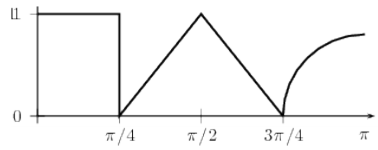


Figure 2: Problem 3:a

Now consider the following multirate processing scheme in which $L(z)$ is an ideal lowpass filter with cutoff frequency $\frac{\pi}{2}$ and $H(z)$ is an ideal highpass filter with cutoff frequency $\frac{\pi}{2}$:

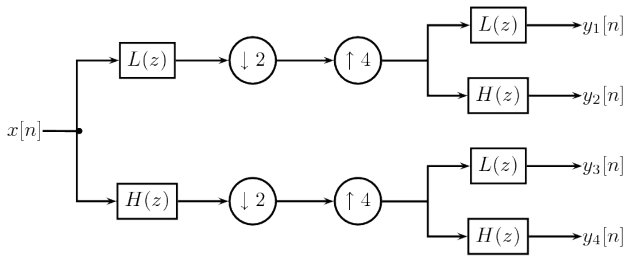
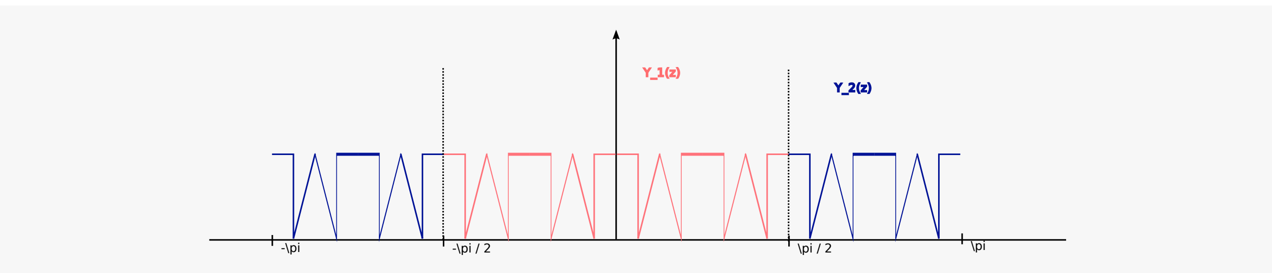
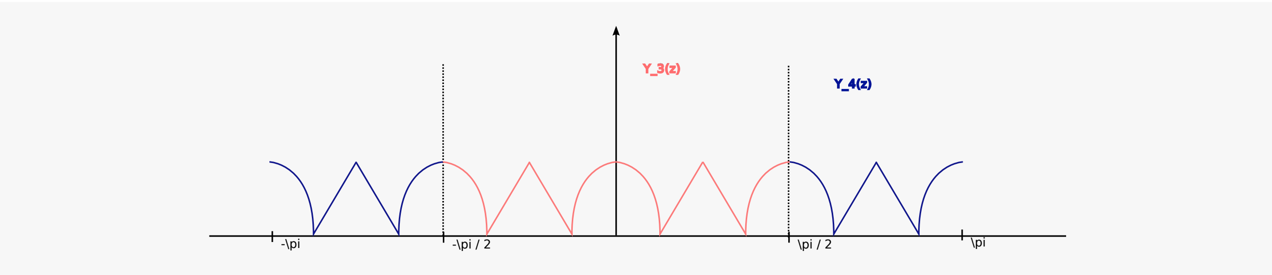


Figure 3: Problem 3:b

Plot the four spectra $Y_1(e^{j\omega})$, $Y_2(e^{j\omega})$, $Y_3(e^{j\omega})$ and $Y_4(e^{j\omega})$.





Problem 4. Consider the input-output relations of the following multirate systems. Remember that, technically, one cannot talk of transfer functions in the case of multirate systems since ampling rate changes are not time invariant. It may happen, though, that by carefully designing the processing chain, this said relation does indeed implement a transfer function.

1. Find the overall transformation operated by the following system:

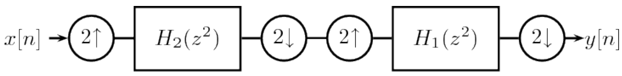


Figure 4: Problem 4:a

2. In the system below, if $H(z) = E_0(z^2) + z^{-1}E_1(z^2)$ for some $E_{0,1}(z)$, prove that $Y(z) = X(z)E_0(z)$.

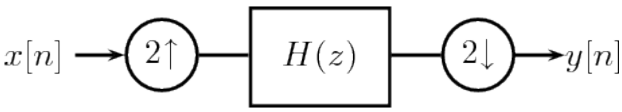


Figure 5: Problem 4:b

3. Let $H(z)$, $F(z)$ and $G(z)$ be filters satisfying

$$H(z)G(z) + H(-z)G(-z) = 2$$

$$H(z)F(z) + H(-z)F(-z) = 0$$

Prove that one of the following systems is unity and the other zero:

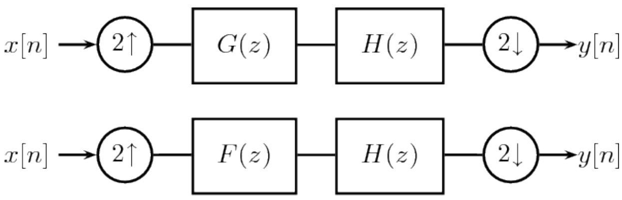
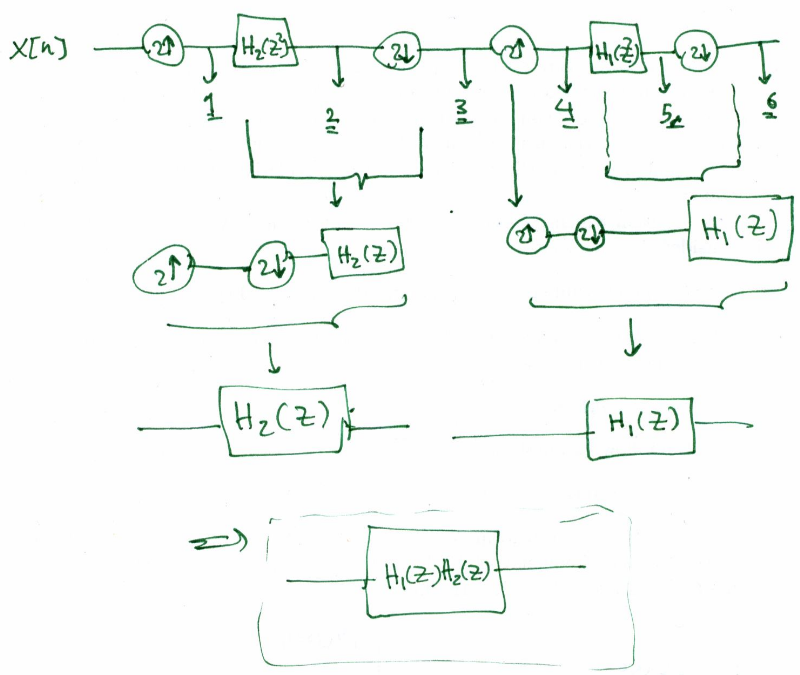


Figure 6: Problem 4:c

1)



2)

$\Rightarrow$ 1 $\longrightarrow$ $X(z^2)$

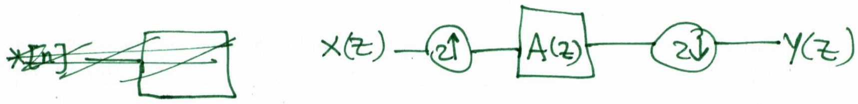
2 $\longrightarrow$ $X(z^2)H(z)$

3 $\longrightarrow$ $\frac{1}{2}\left[X(z^2)H(z)\Big|_{z=z^{1/2}} + X(z^2)H(z)\Big|_{z=-z^{1/2}} \right]$

$= \frac{1}{2}\left[X(z)\left(H(z^{1/2}) + H(-z^{1/2}) \right) \right]$

$= \frac{1}{2} X(z) \left(\frac{2E_0(z) + (z^{1/2} - z^{1/2})E_1(z)}{2} \right) = X(z)E_0(z)$

3) using the same procedure as in part 2) we obtain



$$Y(z) = X(z)\, \tfrac{1}{2}\left[A(z^{1/2}) + A(-z^{1/2})\right]$$

now for the first system $A(z) = G(z)H(z)$

thus $A(z^{1/2}) + A(-z^{1/2}) = G(z^{1/2})\,H(z^{1/2}) + G(-z^{1/2})\,H(-z^{1/2})$

$= 2$ → thus this system is unity

and for the second system similarly we obtain $A(z^{1/2}) + A(-z^{1/2}) = 0$, thus it is zero.

Problem 5. In your grandmothers attic you just found a treasure: a collection of super-rare 78 rpm vinyl jazz records. The first thing you want to do is to transfer the recordings to compact discs, so you can listen to them without wearing out the originals. Your idea is obviously to play the record on a turntable and use an A/D converter to convert the line-out signal into a discrete-time sequence, which you can then burn onto a CD. The problem is, you only have a modern turntable, which plays records at 33 rpm. Since youre a DSP wizard, you know you can just go ahead, play the 78 rpm record at 33 rpm and sample the output of the turntable at 44.1 KHz. You can then manipulate the signal in the discrete-time domain so that, when the signal is recorded on a CD and played back, it will sound right.

Design a system which performs the above conversion. If you need to get on the right track, consider the following:

- Call $s(t)$ the continuous-time signal encoded on the 78 rpm vinyl (the jazz music).
- Call $x(t)$ the continuous-time signal you obtain when you play the record at 33 rpm on the modern turntable.
- Let $x[n] = x(nT_s)$, with $T_s = \frac{1}{44100}$.

Answer the following questions:

1. Express $x(t)$ in terms of $s(t)$.
2. Sketch the Fourier transform $X(j\Omega)$ when $S(j\Omega)$ is as in the following figure. The highest nonzero frequency of $S(j\Omega)$ is $\Omega_{\max} = (2\pi)16,000$ Hz (old records have a smaller bandwidth than modern ones).

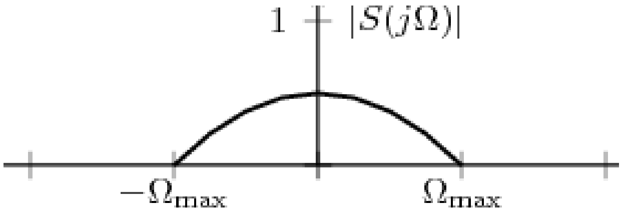


Figure 7: Problem 5

3. Design a system to convert $x[n]$ into a sequence $y[n]$ so that, when you interpolate $y[n]$ to a continuous-time signal $y(t)$ with interpolation period T_s, you obtain $Y(j\Omega) = S(j\Omega)$.
4. What if you had a turntable which plays records at 45 rpm? Would your system be different? Would it be better?