ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

Handout 30 Signal Processing for Communications
Homework 12 May 17, 2010

PROBLEM 1. Consider a real, continuous-time signal $x_c(t)$ with the following spectrum $X_c(j\Omega)$:

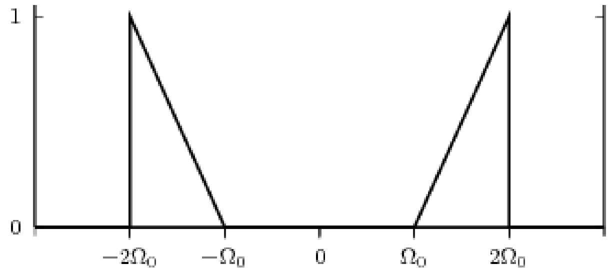

Figure 1: Problem 1

1. What is the bandwidth of the signal? What is the minimum sampling period in order to satisfy the sampling theorem?

2. Take a sampling period $T_s = \frac{\pi}{\Omega_0}$; clearly, with this sampling period, there will be aliasing. Plot the DTFT of the discrete-time signal $x_a[n] = x_c(nT_s)$.

3. Suggest a block diagram to reconstruct $x_c(t)$ from $x_a[n]$.

4. With such a scheme available, we can therefore exploit aliasing to reduce the sampling frequency necessary to sample a bandpass signal. In general, what is the minimum sampling frequency to be able to reconstruct, with the above strategy, a real signal whose frequency support on the positive axis is $[\Omega_0, \Omega_1]$ (with the usual symmetry around zero, of course)?

PROBLEM 2. Consider a discrete-time sequence $x[n]$ with DTFT $X(e^{j\omega})$. Next, consider the continuous-time interpolated signal

$$x_0(t) = \sum_{n=-\infty}^{\infty} x[n] \operatorname{rect}(t-n)$$

i.e. the signal interpolated with a zero-centered zero-order hold and $T = 1$ sec.

1. Express $X_0(j\Omega)$ (the spectrum of $x_0(t)$) in terms of $X(e^{j\omega})$.

2. Compare $X_0(j\Omega)$ to $X(j\Omega)$. We can look at $X(j\Omega)$ as the Fourier transform of the signal obtained from the sinc interpolation of $x[n]$ (always with $T = 1$):

$$x(t) = \sum_{n\in\mathcal{Z}} x[n] \operatorname{sinc}(t-n)$$

Comment on the result: you should point out two major problems.

So, as it appears, interpolating with a zero-order hold introduces a distortion in the interpolated signal with respect to the sinc interpolation in the region $-\pi \leq \Omega \leq \pi$. Furthermore, it makes the signal non-bandlimited outside the region $-\pi \leq \Omega \leq \pi$. The signal $x(t)$ can be obtained from the zero-order hold interpolation $x_0(t)$ as $x(t) = x_0(t) * g(t)$ for some filter $g(t)$.

1. Sketch the frequency response of $g(t)$.
2. Propose two solutions (one in the continuous-time domain, and another in the discrete-time domain) to eliminate or attenuate the distortion due to the zero-order hold. Discuss the advantages and disadvantages of each.

Problem 3. Consider the local interpolation scheme of the previous exercise but assume that the characteristic of the interpolator is the following:

$$I(t) = \begin{cases} 1 - 2|t| & |t| \leq 0.5 \\ 0 & \text{otherwise} \end{cases}$$

This is a triangular characteristic with the same support as the zero-order hold. If we pick an interpolation interval T_s and interpolate a given discrete-time signal $x[n]$ with $I(t)$, we obtain a continuous-time signal:

$$x(t) = \sum_n x[n] I(\frac{t - nT_s}{T_s})$$

which looks like this:

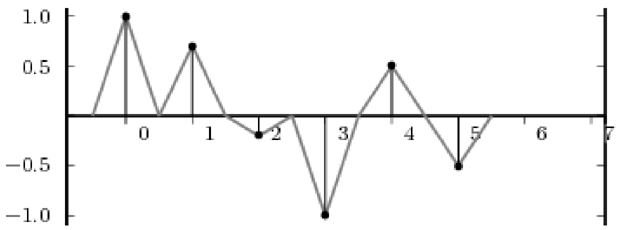

Figure 2: Problem 3

Assume that the spectrum of $x[n]$ between $-\pi$ and π is

$$X(e^{j\omega}) = \begin{cases} 1 & |w| \leq \frac{2\pi}{3} \\ 0 & \text{otherwise} \end{cases}$$

(with the obvious 2π-periodicity over the entire frequency axis).

1. Compute and sketch the Fourier transform $I(j\Omega)$ of the interpolating function $I(t)$. (Recall that the triangular function can be expressed as the convolution of $\text{rect}(\frac{t}{2})$ with itself).
2. Sketch the Fourier transform $X(j\Omega)$ of the interpolated signal $x(t)$; in particular, clearly mark the Nyquist frequency $\Omega_N = \frac{\pi}{T_s}$.
3. The use of $I(t)$ instead of a sinc interpolator introduces two types of errors: briefly describe them.

4. To eliminate the error in the baseband $[-\Omega_N, \Omega_N]$ we can pre-filter the signal $x[n]$ with a filter $h[n]$ before interpolating with $I(t)$. Write the frequency response of the discrete-time filter $H(e^{j\omega})$.

Problem 4. Consider an N-periodic discrete-time signal $\tilde{x}[n]$, with N an even number, and let $\tilde{X}[k]$ be its DFS:

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}nk}, \qquad k \in \mathcal{Z}$$

Let $\tilde{Y}[m] = \tilde{X}[2m]$, i.e. a subsampled version of the DFS coefficients; clearly this defines a $(\frac{N}{2})$-periodic sequence of DFS coefficients. Now consider the $(\frac{N}{2})$-point inverse DFS of $\tilde{Y}[m]$ and call this $(\frac{N}{2})$-periodic signal $\tilde{y}[n]$:

$$\tilde{y}[n] = \frac{2}{N} \sum_{n=0}^{\frac{N}{2}-1} \tilde{Y}[k] e^{-j\frac{2\pi}{\frac{N}{2}}nk}, \qquad n \in \mathcal{Z}$$

Express $\tilde{y}[n]$ in terms of $\tilde{x}[n]$ and describe in a few words what has happened to $\tilde{x}[n]$ and why.

Problem 5. Consider a zero-mean white random process $X[n]$ with autocorrelation function $r_X[m] = \sigma^2\delta[m]$. The process is filtered with leaky integrator ($H(z) = \frac{1-\lambda}{1-\lambda z^{-1}}$) producing the signal $Y[n] = X[n] * h[n]$. Compute the values of the input-output cross-correlation from $n = -3$ to $n = 3$.

Problem 6. You have a continuous-time signal (for example, a music source), which you want to store on a digital medium such as a memory card. Assume, for simplicity, that the signal has already been sampled (but not quantized) with a sampling frequency $F_s = 32000$ Hz with no aliasing. Assume further that the sampled signal can be modeled as a white process $x[n]$ with power spectral density

$$P_x(e^{j\omega}) = \sigma_x^2$$

and that the pdf of each sample is uniform on the $[-1, 1]$ interval.

Now you need to quantize and store the signal. Your constraints are the following:

1. You want to store exactly one seconds worth of the input signal.
2. The capacity of the memory card is 32000 bytes.
3. You can either use a 8-bit quantizer (Quantizer A) or a 16-bit quantizer (Quantizer B). Both quantizers are uniform over the $[-1, 1]$ interval.

You come up with two possible schemes:

- You quantize the samples with quantizer A and store them on the card.
- You first downsample the signal by 2 (with lowpass filtering) and then use quantizer B.

Clearly both schemes fulfill your constraints. Question: which is the configuration which minimizes the overall mean square error between the original signal and the digitized signal? Show why. As a guideline, note that the MSE will be composed of two independent parts: the one introduced by the quantizers and, for the second scheme, the one which is introduced by the lowpass filter before the downsampler. For the quantizer error, you can assume that the downsampled process still remains a uniform, i.i.d. process.