Midterm - SPC

Midterm

P1 -

a) $$H(z) = \frac{(1-z^{-1})(1-2z^{-1})}{(1-0.25z^{-1})(1-0.16z^{-2})}$$

$$G'(z) = \frac{1}{(1-0.25z^{-1})(1-0.16z^{-2})}$$

$$H(z) = (1-3z^{-1}+2z^{-2})\, G(z)$$

$$\Rightarrow h[n] = g'[n] - 3g'[n-1] + 2g'[n-2]$$

$$G'(z) = \frac{a}{1-0.25z^{-1}} + \frac{b}{1-0.4z^{-1}} + \frac{c}{1+0.4z^{-2}}$$

$$a = G'(z) \times (1-0.25z^{-1})\Big|_{z=0.25} = \frac{1}{1-0.16\times(0.25)^{-2}}$$

$$b = G'(z) \times (1-0.4z^{-1})\Big|_{z=0.4} = \frac{1}{\left(1-\frac{0.25}{0.4}\right)\left(1+\frac{0.4}{0.4}\right)}$$

$$c = G'(z) \times (1+0.4z^{-1})\Big|_{z=-0.4} = \frac{1}{\left(1+\frac{0.25}{0.4}\right)\left(1+\frac{0.4}{0.4}\right)}$$

$$g'[n] = a\cdot(0.25)^n u[n] + b\,(0.4)^n u[n] + c\,(-0.4)^n u[n].$$

Problem 1)

(b)

(i) $$G(z) = \frac{(1-0.25z^{-1})(1-0.16z^{-2})}{(1-z^{-1})(1-2z^{-1})}$$

(ii) $$G(z) = \frac{1-0.25z^{-1}-0.16z^{-2}+0.04z^{-3}}{(1-z^{-1})(1-2z^{-1})}$$

$$K(z) = \frac{1}{(1-z^{-1})(1-2z^{-1})}$$

$$g[n] = K[n] - 0.25\,K[n-1] - 0.16\,K[n-2] + 0.04\,K[n-3]$$

$$K[z] = \frac{a}{1-z^{-1}} + \frac{b}{1-2z^{-1}}$$

$$a = K[z]\,(1-z^{-1})\Big|_{z=1} = -1$$

$$b = K[z]\,(1-2z^{-1})\Big|_{z=2} = 2$$

$$K[n] = -u[n] + 2\cdot 2^{n}\,u[n]$$

As one can see, this $K[n]$ is <u>not</u> stable, neither is $g[n]$.

Problem 2)

S1)

Put $x[n]$ into even slots

and $y[n]$ into odd slots.

$X[K]$: take the averag of first half ~~and~~ and the second half.

$Y[K]$: take the differenc of first half and the second half and divide by W_n^K.

second solution)

Load $x[n]$ in Real part

and $y[n]$ in imaginary part

and FFT of N-Point is enough.

Problem 3)

$$z[n] = x[n] * y[n] \quad , \quad z[n] = \frac{1}{2\pi}\int_0^{2\pi} Z(e^{j\omega}) e^{j\omega n} d\omega$$

$$\Rightarrow \quad Z(e^{j\omega}) = X(e^{j\omega}) \cdot Y(e^{j\omega})$$

$$X(e^{j\omega}) = \frac{1}{2}\left[e^{j\frac{\pi}{17}} \delta(\omega - \frac{\pi}{49}) + e^{-j\frac{\pi}{17}} \cdot \delta(\omega + \frac{\pi}{49}) \right]$$

$$Y(e^{j\omega}) = \sum_{n=-49}^{49} \frac{(49-n)}{49} e^{-j\omega n}$$

$$z[n] = \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{n'=-49}^{49} \frac{(49-n')}{49} e^{-j\omega n'} \right)$$

$$\cdot \frac{1}{2}\left[e^{j\frac{\pi}{17}} \delta(\omega - \frac{\pi}{49}) + e^{-j\frac{\pi}{17}} \delta(\omega + \frac{\pi}{49}) \right]$$

$$\cdot \, e^{j\omega n} \, d\omega$$

$$= \frac{1}{2\pi} \times \frac{1}{2} \times e^{j\frac{\pi}{17}} \times \underbrace{\left(\sum_{n'=-49}^{49} \frac{49-n'}{49} e^{-j\frac{\pi}{49} n'} \right)}_{a} \times e^{j\frac{\pi}{49} n} \times 2\pi$$

$$+ \frac{1}{2\pi} \times \frac{1}{2} \times e^{-j\frac{\pi}{17}} \times \underbrace{\left(\sum_{n'} \frac{49-n'}{49} e^{j\frac{\pi}{49} n'} \right)}_{a^*} \times e^{-j\frac{\pi}{49} n} \times 2\pi$$

$$= \frac{1}{2} e^{j\frac{\pi}{17}} a \, e^{j\frac{\pi}{49} n}$$

$$+ \frac{1}{2} e^{j\frac{\pi}{17}} a^* \, e^{-j\frac{\pi}{49} n}$$

Now, we only need to comput a.

$$a = \sum_{n'=0}^{98} \frac{n'}{49} \cdot e^{-j\frac{\pi}{49}(-n'+49)}$$

$$= \frac{e^{-j\pi}}{49} \sum_{n'=0}^{98} n' \, e^{j\frac{\pi}{49}n'}$$

We know

$$\sum_{n=0}^{K} \alpha^n = \frac{1-\alpha^{K+1}}{1-\alpha}$$

$$\frac{d}{d\alpha} : \sum_{n=0}^{K} n \alpha^{n-1} = \frac{-(K+1)\alpha^{K}(1-\alpha) + (1-\alpha^{K+1})}{(1-\alpha)^2}$$

$$\Rightarrow \sum_{n=0}^{K} n \alpha^{n} = \frac{-(K+1)\alpha^{K+1} + (K+1)\alpha^{K+2} + 1 - \alpha^{K+1}}{(1-\alpha)^2}$$

$$= \frac{1 - (K+2)\alpha^{K+1} + (K+1)\alpha^{K+2}}{(1-\alpha)^2}$$

so,

$$a = \frac{e^{-j\pi}}{49} \cdot \frac{1 - 100 \cdot e^{j\frac{\pi}{49}\times 99} + 99 \cdot e^{j\frac{\pi}{49}\times 100}}{(1 - e^{j\frac{\pi}{49}})^2}$$

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE
School of Computer and Communication Sciences

PROBLEM 4 (a) Consider $x[n]$ and $y[n]$. As mentioned in the question, $r[n]$ is the circular convolution of these two signals:

$$r[n] = \sum_{k=0}^{19} y[k]x[(n-k) \mod N] \tag{1}$$

Let the linear convolution of $x[n]$ and $y[n]$ be $z[n]$:

$$z[n] = \sum_{k=0}^{19} y[k]x[n-k] \tag{2}$$

So we are looking for all n for which $((n-k) \mod N) = n-k \ \forall 0 \leq k \leq 19$ and this is true for all $7 \leq n \leq 19$.

(b)

$$z[n] = \sum_{k=0}^{19} x[k]y[n-k] \tag{3}$$

At each time n, reflect $y[k]$ with respect to $k = 0$, shift it n units, multiply the obtained signal ($y[n-k]$) with $x[k]$, and sum them for each n over all k. one should further note that $z[n] = 0$ for $n < 0$ and $n > 19 + 7$. We draw the convolution in detail for $n = 0, 1, 2, 3, 4, 5, 6$ for example (fig 1). Thus $z[n]$ would be as shown in figure 2

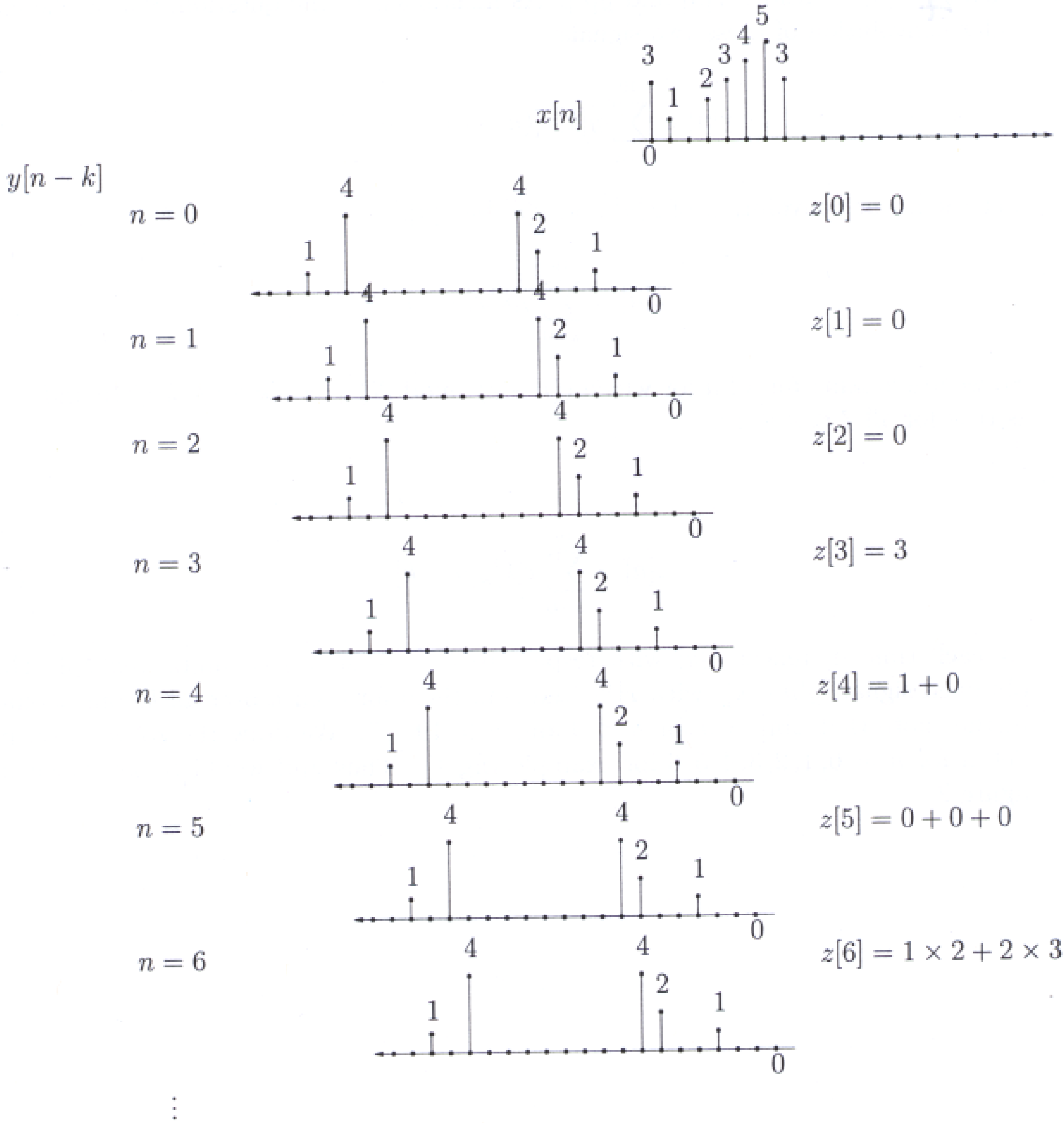

Figure 1: Convolution in problem 4

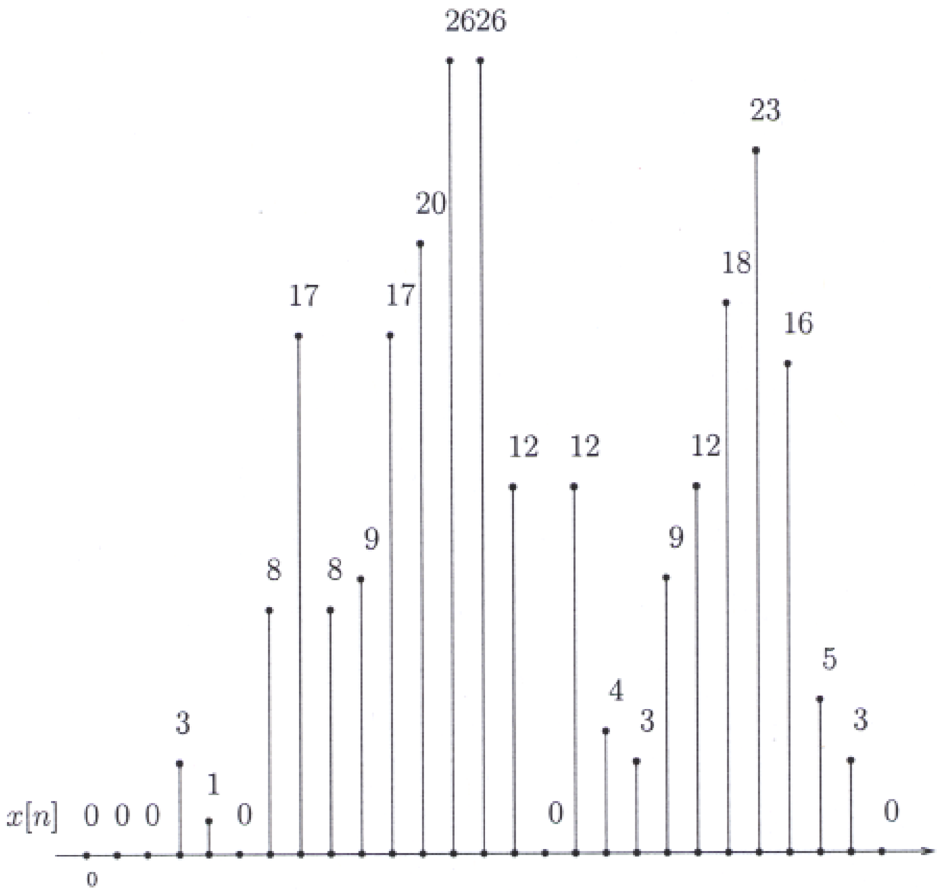

Figure 2: $z[n]$ obtained in problem 4

Problem 5)

(a) $$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$\underline{n' = 3-n}$

$$FT\{x^*[3-n]\} = \sum_{n=-\infty}^{\infty} x^*[3-n]\, e^{-j\omega n}$$

$$= \sum_{n'=-\infty}^{\infty} x^*[n']\, e^{-j\omega(3-n')}$$

$$= e^{-j3\omega} \cdot \sum_{n'=-\infty}^{\infty} x^*[n']\, e^{j\omega n'}$$

$$= e^{-j3\omega} \cdot X^*(e^{j\omega})$$

$\omega' = -3\omega + \omega_0$

(b) $$\int_{-\infty}^{\infty} e^{j\omega t}\, \delta(-3\omega + \omega_0)\, d\omega =$$

$$= \int_{\infty}^{-\infty} e^{j\left(\frac{\omega' - \omega_0}{-3}\right)t} \cdot \delta(\omega') \cdot \frac{d\omega'}{-3}$$

$$= \int_{-\infty}^{\infty} \frac{1}{3}\, e^{j\left(\frac{\omega' - \omega_0}{-3}\right)t}\, \delta(\omega')\, d\omega'$$

$$= \frac{1}{3}\, e^{j\frac{\omega_0}{3}t}$$

Problem 5)

c) $y[n] = \cos\left(\frac{\pi}{8} n\right) + \cos\left(\frac{2\pi}{8} n\right) + \cos\left(\frac{3\pi}{8} n\right]$

The rest are filtered out by the filter.

(d) $\cos(2x) - \cos(3x) =$

$-4\cos^3(x) + 2\cos^2(x) + 3\cos(x) - 1$