Solutions 7

1. a) We have:

$$\mathrm{Re}\, g_\mu(u+iv) = \int_{\mathbb{R}} \frac{x-u}{(x-u)^2+v^2}\, d\mu(x) \quad \text{and} \quad \mathrm{Im}\, g_\mu(u+iv) = \int_{\mathbb{R}} \frac{v}{(x-u)^2+v^2}\, d\mu(x).$$

b) No proof required: the analyticity of g_μ on $\mathbb{C}\backslash\mathbb{R}$ follows from the analyticity of $z \mapsto \frac{1}{x-z}$ on $\mathbb{C}\backslash\mathbb{R}$ and the use of the dominated convergence theorem.

c) If $v > 0$, then $\mathrm{Im}\, g_\mu(u+iv)$ is clearly positive by the above formula.

d) We have:

$$v^2\, |g_\mu(iv)|^2 = \left(\int_{\mathbb{R}} \frac{v(x-u)}{(x-u)^2+v^2}\, d\mu(x)\right)^2 + \left(\int_{\mathbb{R}} \frac{v^2}{(x-u)^2+v^2}\, d\mu(x)\right)^2.$$

By the dominated convergence theorem, the first term on the right-hand side converges to 0 as $v \to +\infty$ and the second term converges to 1.

e) This is a straightforward computation.

2. a) We have

$$\begin{aligned}
\frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\int_a^b \mathrm{Im}\, g_\mu(x+i\varepsilon)\, dx &= \frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\int_a^b \left(\int_{\mathbb{R}} \frac{\varepsilon}{(y-x)^2+\varepsilon^2}\, d\mu(y)\right) dx \\
&= \frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}} \left(\int_a^b \frac{\varepsilon}{(y-x)^2+\varepsilon^2}\, dx\right) d\mu(y) \\
&= \frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}} \mathrm{arctg}\left(\frac{y-x}{\varepsilon}\right)\Bigg|_{x=a}^{x=b} d\mu(y).
\end{aligned}$$

Since

$$\lim_{\varepsilon\downarrow 0} \mathrm{arctg}\left(\frac{y-x}{\varepsilon}\right)\Bigg|_{x=a}^{x=b} = \begin{cases} \pi, & \text{if } a<y<b, \\ \frac{\pi}{2}, & \text{if } y=a \text{ or } b \\ 0, & \text{otherwise} \end{cases} = \pi\,(1_{]a,b[}(y) + \frac{1}{2}\, 1_{\{a,b\}}(y)),$$

we conclude by the dominated convergence theorem that

$$\frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\int_a^b \mathrm{Im}\, g_\mu(x+i\varepsilon)\, dx = \mu(\,]a,b[\,),$$

at any $a<b$ continuity points of F_μ.

b) Assuming that μ has a pdf p_μ, the very same computation as above leads to

$$\begin{aligned}
\frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\int_a^b \mathrm{Im}\, g_\mu(x+i\varepsilon)\, dx &= \frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\int_a^b \left(\int_{\mathbb{R}} \frac{\varepsilon}{(y-x)^2+\varepsilon^2}\, p_\mu(y)\, dy\right) dx \\
&= \int_{\mathbb{R}} (1_{]a,b[}(y) + \frac{1}{2}\, 1_{\{a,b\}}(y))\, p_\mu(y)\, dy = \int_a^b p_\mu(y)\, dy.
\end{aligned}$$

3. a) We have for $z = x + iy$,

$$g_0(z) = \frac{1}{x_0 - iy_0 - x - iy} = \frac{x_0 - x + i(y_0 + y)}{(x_0 - x)^2 + (y_0 + y)^2},$$

so

$$\mathrm{Im}(g_0(z)) = \frac{(y_0 + y)}{(x_0 - x)^2 + (y_0 + y)^2}.$$

Let us first consider the case where $y_0 = 0$. Then the result of Exercise 2, part a), tells us that for any $a < x_0 < b$,

$$\begin{aligned}\mu(\,]a,b[\,) &= \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_a^b \frac{\varepsilon}{(x_0 - x)^2 + \varepsilon^2}\, dx \\ &= \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \operatorname{arctg}\left(\frac{x - x_0}{\varepsilon}\right)\Bigg|_{x=a}^{x=b} = \frac{1}{\pi}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 1.\end{aligned}$$

So that $\mu(\,]a,b[\,) = 1$ for all $a < x_0 < b$, i.e. $\mu = \delta_{x_0}$ and the moments of μ are $m_k = x_0^k$.

For the case $y_0 > 0$, we have

$$p_\mu(x) = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \frac{y_0 + \varepsilon}{(x_0 - x)^2 + (y_0 + \varepsilon)^2} = \frac{1}{\pi} \frac{y_0}{(x_0 - x)^2 + y_0^2},$$

which is the Cauchy distribution with parameters x_0 and y_0. This distribution has no finite moments, but notice that x_0 is closely related to its "mean" and that $1/y_0$ is a measure of how spread the distribution is.

b) The solution of the equation is

$$g_\pm(z) = -\frac{z}{2} \pm \sqrt{\frac{z^2}{4} - 1}$$

and for $\mathrm{Im} z > 0$, only g_+ satsifies $\mathrm{Im} g_+(z) > 0$. Therefore, by Exercise 2, part b), we have

$$\begin{aligned}p_\mu(x) &= \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \mathrm{Im}(g_+(z)) = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \left(-\frac{\varepsilon}{2} + \mathrm{Im}\left(\sqrt{\frac{(x+i\varepsilon)^2}{4} - 1}\right)\right) \\ &= \frac{1}{\pi} \mathrm{Im}\left(\sqrt{\frac{x^2}{4} - 1}\right) = \frac{1}{2\pi} \sqrt{4 - x^2}\, 1_{\{|x| \le 2\}}.\end{aligned}$$

c) The solution of the equation is

$$g_\pm(z) = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{1}{z}}$$

and for $\mathrm{Im} z > 0$, only g_+ satsifies $\mathrm{Im} g_+(z) > 0$. Therefore,

$$\begin{aligned}p_\mu(x) &= \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \mathrm{Im}(g_+(z)) = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \mathrm{Im}\left(\sqrt{\frac{1}{4} - \frac{1}{x + i\varepsilon}}\right) \\ &= \frac{1}{\pi} \mathrm{Im}\left(\sqrt{\frac{1}{4} - \frac{1}{x}}\right) = \frac{1}{\pi} \sqrt{\frac{1}{x} - \frac{1}{4}}\, 1_{\{0 < x \le 4\}}.\end{aligned}$$

4. Let us first mention that if $|\rho| = 1$, the series

$$\sum_{l \in \mathbb{Z}} |\rho|^{|l|} \tag{1}$$

is infinite, so the Grenander-Szegö theorem does not apply. We therefore have to perform a separate analysis in this case.

If $\rho = +1$, then the matrix $T^{(n)}$ is the all-one matrix; it therefore has one eigenvalue equal to n (corresponding to the all-one eigenvector) and all others equal to zero (corresponding to the remaining $n-1$ eigenvectors, orthogonal to the all-one vector). We therefore have for any bounded continuous function f:

$$\frac{1}{n} \sum_{j=1}^{n} f(\lambda_j^{(n)}) = \frac{1}{n} f(n) + \frac{n-1}{n} f(0) \underset{n \to \infty}{\to} f(0).$$

i.e., the empirical eigenvalue distribution of $T^{(n)}$ converges weakly to the Dirac mass δ_0 at point $x = 0$.

If $\rho = -1$, then $T^{(n)}$ is the matrix with alternating ± 1; it also has one eigenvalue equal to n (corresponding to the alternating ± 1 eigenvector), all other eigenvalues being 0. So the empirical eigenvalue distribution of $T^{(n)}$ converges weakly to δ_0 also in this case.

In the case where $\rho \in]-1, +1[$, the series (1) is finite, so the Grenander-Szegö theorem applies. Let us compute the function g:

$$\begin{aligned} g(x) &= \sum_{l \in \mathbb{Z}} \rho^{|l|} e^{ilx} = 1 + \sum_{l=1}^{\infty} \rho^l \, (e^{ilx} + e^{-ilx}) \\ &= 1 + \sum_{l=1}^{\infty} \left((\rho e^{ix})^l + (\rho e^{-ix})^l \right) = \frac{1}{1 - \rho e^{ix}} + \frac{1}{1 - \rho e^{-ix}} - 1 \\ &= \frac{2(1 - \mathrm{Re}(\rho e^{ix}))}{|1 - \rho e^{ix}|^2} - 1 = \frac{1 - \rho^2}{1 + \rho^2 - 2\rho \cos x}. \end{aligned}$$

By the theorem, we have for any bounded continuous function f:

$$\frac{1}{n} \sum_{j=1}^{n} f(\lambda_j^{(n)}) \underset{n \to \infty}{\to} \frac{1}{\pi} \int_0^{\pi} f(g(x)) \, dx.$$

Note that

$$\frac{1 - |\rho|}{1 + |\rho|} \leq g(x) \leq \frac{1 + |\rho|}{1 - |\rho|}$$

and that these two bounds are those of the limiting spectrum of $T^{(n)}$. We now make the change of variable $y = g(x)$, so that

$$dy = -\frac{1 - \rho^2}{(1 + \rho^2 - 2\rho \cos x)^2} \, 2\rho \sin x \, dx.$$

Inverting this relation (using the fact that $\sin x = \sqrt{1 - \cos^2 x}$), we obtain:

$$dx = -\frac{1 - \rho^2}{y^2 \, 2\rho \sin x} \, dx = -\frac{\sqrt{1 - \rho^2}}{y \sqrt{\rho^2 (y+1)^2 - (y-1)^2}} \, dy,$$

so that

$$\frac{1}{n} \sum_{j=1}^{n} f(\lambda_j^{(n)}) \underset{n \to \infty}{\to} \int_{\frac{1-|\rho|}{1+|\rho|}}^{\frac{1+|\rho|}{1-|\rho|}} f(y) \, \frac{\sqrt{1 - \rho^2}}{\pi y \sqrt{\rho^2 (y+1)^2 - (y-1)^2}} \, dy.$$