Solutions 5

1. a) First note that we have the following equivalent definitions:

$$\|A\|_1 = \sup_{x\in\mathbb{C}^n:\|x\|=1} \|Ax\| \quad \text{and} \quad \|A\|_2 = \sqrt{\frac{1}{n}\sum_{j,k=1}^n |a_{jk}|^2}.$$

By the Cauchy-Schwarz inequality, we obtain:

$$\left|\frac{1}{n}\mathrm{Tr}(A)\right| = \left|\frac{1}{n}\sum_{j=1}^n a_{jj}\right| \leq \frac{1}{n}\sqrt{n\sum_{j=1}^n |a_{jj}|^2} \leq \sqrt{\frac{1}{n}\sum_{j,k=1}^n |a_{jk}|^2} = \|A\|_2.$$

For $k \in \{1,\ldots,n\}$, we denote by $\delta^{(k)}$ the column vector whose components are given by $\delta_j^{(k)} = 1$ if $j = k$, 0 otherwise. We then have

$$\|A\|_1^2 \geq \max_{k\in\{1,\ldots,n\}} \|A\delta^{(k)}\|^2 \geq \frac{1}{n}\sum_{k=1}^n \|A\delta^{(k)}\|^2 = \frac{1}{n}\sum_{j,k=1}^n |a_{jk}|^2 = \|A\|_2^2.$$

Next, we see that since $\|Ax\| \leq \|A\|_1\,\|x\|$,

$$\|AB\|_1 = \sup_{x\in\mathbb{C}^n:\|x\|=1} \|ABx\| \leq \sup_{x\in\mathbb{C}^n:\|x\|=1} \|A\|_1\,\|Bx\| = \|A\|_1\,\|B\|_1.$$

Finally, let us denote by $b^{(k)}$ the k-th column vector of the matrix B (*i.e.*, $b_j^{(k)} = b_{jk}$); we have

$$\begin{aligned}\|AB\|_2^2 &= \frac{1}{n}\sum_{j,k=1}^n \left|\sum_{l=1}^n a_{jl}b_{lk}\right|^2 = \frac{1}{n}\sum_{j,k=1}^n \left|(Ab^{(k)})_j\right|^2 = \frac{1}{n}\sum_{k=1}^n \|Ab^{(k)}\|^2 \\ &\leq \frac{1}{n}\sum_{k=1}^n \|A\|_1^2\,\|b^{(k)}\|^2 = \|A\|_1^2\,\|B\|_2^2.\end{aligned}$$

b) Without any assumption on A, we know that $\beta_j \geq 0$ for all j. If A is Hermitian, then $\alpha_j \in \mathbb{R}$ and $\beta_j = \alpha_j^2$ (provided that we have ordered the eigenvalues correspondingly).

Now, another possible equivalent definition for $\|A\|_1$ is

$$\|A\|_1^2 = \sup_{x\in\mathbb{C}^n:\|x\|=1} x^*A^*Ax,$$

and A^*A is diagonalizable (because it is Hermitian), so $A^*A = U^*DU$ for some unitary matrix U and $D = \mathrm{diag}(\beta_1,\ldots,\beta_n)$. This implies that

$$\|A\|_1^2 = \sup_{x\in\mathbb{C}^n:\|x\|=1} x^*U^*DUx = \sup_{x\in\mathbb{C}^n:\|x\|=1} x^*Dx = \sup_{x\in\mathbb{C}^n:\|x\|=1} \sum_{j=1}^n \beta_j|x_j|^2 = \max_{j\in\{1,\ldots,n\}} \beta_j.$$

Similarly,

$$\|A\|_2^2 = \frac{1}{n}\mathrm{Tr}(A^*A) = \frac{1}{n}\mathrm{Tr}(U^*DU) = \frac{1}{n}\mathrm{Tr}(D) = \frac{1}{n}\sum_{j=1}^n \beta_j.$$

Finally, using the fact that any square matrix A is similar to a Jordan matrix, *i.e.*, that there exists an invertible matrix S such that

$$A = SJS^{-1},$$

where J is an upper-triangular matrix whose main diagonal is composed of the eigenvalues $\alpha_1, \ldots, \alpha_n$ of A, we deduce that

$$A^m = SJ^mS^{-1}, \quad \text{therefore,} \quad \frac{1}{n}\mathrm{Tr}(A^m) = \frac{1}{n}\mathrm{Tr}(J^m) = \frac{1}{n}\sum_{j=1}^{n} \alpha_j^m.$$

c) Since $T^{(n)}$ and $C^{(n)}$ are both Hermitian, we deduce from part 2 that

$$\|T^{(n)}\|_1 \leq \max_{j\in\{1,\ldots,n\}} \left|\lambda_j^{(n)}\right| \quad \text{and} \quad \|C^{(n)}\|_1 \leq \max_{j\in\{1,\ldots,n\}} \left|\mu_j^{(n)}\right|.$$

Using then Geršgorin discs' argument, we deduce that

$$\|T^{(n)}\|_1 \leq 4 \quad \text{and} \quad \|C^{(n)}\|_1 \leq 4.$$

NB: there are many ways to prove these two inequalities! (in particular, one could use ex. 2 below)

The third inequality is the result of a direct computation:

$$\|T^{(n)} - C^{(n)}\|_2^2 = \frac{1}{n}(1+1) = \frac{2}{n}.$$

From these inequalities and the above results, we deduce finally that

$$\begin{aligned}
\left|\frac{1}{n}\sum_{k=1}^{n}(\lambda_k^{(n)})^m - \frac{1}{n}\sum_{k=1}^{n}(\mu_k^{(n)})^m\right| &= \left|\frac{1}{n}\mathrm{Tr}((T^{(n)})^m) - \frac{1}{n}\mathrm{Tr}((C^{(n)})^m)\right| \\
&\leq \|(T^{(n)})^m - (C^{(n)})^m\|_2 \leq \sum_{j=1}^{m} \|(T^{(n)})^{j-1}\,(T^{(n)} - C^{(n)})\,(C^{(n)})^{m-j}\|_2 \\
&\leq \sum_{j=1}^{m} \|(T^{(n)})^{j-1}\|_1\,\|T^{(n)} - C^{(n)}\|_2\,\|(C^{(n)})^{m-j}\|_1 \\
&\leq \sqrt{\frac{2}{n}}\sum_{j=1}^{m} \|T^{(n)}\|_1^{j-1}\,\|C^{(n)}\|_1^{m-j} \leq \sqrt{\frac{2}{n}}\sum_{j=1}^{m} 4^{m-1} \underset{n\to\infty}{\to} 0.
\end{aligned}$$

2. We have

$$\begin{aligned}\lambda^{(n)}\, u^* u &= u^* T^{(n)} u = \sum_{j,k=1}^{n} t_{k-j}\, u_j\, \overline{u_k} = \frac{1}{2\pi}\int_0^{2\pi} g(x)\, \left(\sum_{j,k=1}^{n} e^{-i(k-j)x} u_j\, \overline{u_k}\right)\, dx \\ &= \frac{1}{2\pi}\int_0^{2\pi} g(x)\, \left|\sum_{j=1}^{n} e^{-ijx}\, u_j\right|^2\, dx.\end{aligned}$$

Similarly (set $t_0 = 1$ and $t_l = 0$ for all $l \neq 0$ in the above formula), we have

$$u^* u = u^* I u = \frac{1}{2\pi}\int_0^{2\pi} \left|\sum_{j=1}^{n} e^{-ijx}\, u_j\right|^2\, dx.$$

From the above two equations, we easily deduce that

$$\inf_{x\in[0,2\pi]} g(x) \leq \lambda^{(n)} \leq \sup_{x\in[0,2\pi]} g(x).$$