Solutions 4

a) We have

$$\prod_{j<k}\left(e^{i\theta_j}-e^{i\theta_k}\right)=\det\begin{pmatrix} 1 & \ldots & 1 \\ e^{i\theta_1} & \ldots & e^{i\theta_n} \\ & \ldots & \\ e^{i(n-1)\theta_1} & \ldots & e^{i(n-1)\theta_n}\end{pmatrix}=\det\left(\left\{e^{i(j-1)\theta_k}\right\}\right).$$

Using the definition of K, we therefore obtain

$$\begin{aligned} p(\theta_1,\ldots,\theta_n) &= C_n\left|\det\left(\left\{e^{i(j-1)\theta_k}\right\}\right)\right|^2 = C_n\,\det\left(\left\{e^{i(j-1)\theta_k}\right\}^*\left\{e^{i(j-1)\theta_k}\right\}\right) \\ &= C_n\,\det\left(\{K(\theta_j,\theta_k)\}\right). \end{aligned}$$

b) Property (0) is clear, property (i) follows from the fact that $K(\theta,\theta)=\sum_{l=0}^{n-1}1=n$ and property (ii) follows from

$$\int_0^{2\pi}K(\theta,\varphi)\,K(\varphi,\psi)\,d\varphi=\sum_{l,m=0}^{n-1}e^{il\theta}\,\overline{e^{im\psi}}\int_0^{2\pi}\overline{e^{il\varphi}}\,e^{im\varphi}\,d\varphi=2\pi\sum_{l=0}^{n-1}e^{il\theta}\,\overline{e^{il\psi}}=2\pi\,K(\theta,\psi),$$

c) By expanding the determinant with respect to the m^{th} column, we obtain

$$\begin{aligned} &\int_0^{2\pi}D_m(\theta_1,\ldots,\theta_m)\,d\theta_m \\ &= \sum_{l=1}^{m}(-1)^{l+m}\int_0^{2\pi}K(\theta_l,\theta_m)\det\left(\{K(\theta_j,\theta_k\}_{j\neq l,k\neq m}\right)\,d\theta_m \\ &= \int_0^{2\pi}K(\theta_m,\theta_m)\,d\theta_m\,D_{m-1}(\theta_1,\ldots,\theta_{m-1})+\sum_{l=1}^{m-1}(-1)^{l+m}\int_0^{2\pi}K(\theta_l,\theta_m) \\ &\quad\cdot\left(\sum_{l'=1}^{m-1}(-1)^{l'+m-1}K(\theta_m,\theta_{l'})\,\det\left(\{K(\theta_j,\theta_k\}_{j\neq l,m;k\neq l',m}\right)\right)\,d\theta_m, \end{aligned}$$

where we have expanded each sub-determinant in the sum with respect to the $(m-1)^{th}$ row (which is the m^{th} row of the original matrix).

Using properties (i) and (ii) of part b), we therefore obtain

$$\begin{aligned} &\int_0^{2\pi}D_m(\theta_1,\ldots,\theta_m)\,d\theta_m \\ &= 2\pi n\,D_{m-1}(\theta_1,\ldots,\theta_{m-1})-2\pi\sum_{l,l'=1}^{m-1}(-1)^{l+l'}K(\theta_l,\theta_{l'})\det\left(\{K(\theta_j,\theta_k\}_{j\neq l,m;k\neq l',m}\right) \\ &= 2\pi n\,D_{m-1}(\theta_1,\ldots,\theta_{m-1})-2\pi\sum_{l=1}^{m-1}D_{m-1}(\theta_1,\ldots,\theta_{m-1}) \\ &= 2\pi\,(n-m+1)\,D_{m-1}(\theta_1,\ldots,\theta_{m-1}), \end{aligned}$$

where we have again used the formula for the determinant (in the other direction).

d) By the formula found for $p(\theta_1,\ldots,\theta_n)$ in part a) and repeated applications of Mehta's lemma (or induction), we find that

$$\begin{aligned} p(\theta,\varphi) &= C_n \int_{[0,2\pi]^{n-2}} D_n(\theta,\varphi,\theta_3,\ldots,\theta_n)\, d\theta_3\cdots d\theta_n \\ &= C_n\, 2\pi \int_{[0,2\pi]^{n-3}} D_{n-1}(\theta,\varphi,\theta_3,\ldots,\theta_{n-1})\, d\theta_3\cdots d\theta_{n-1} \\ &= \ldots = C_{n,2}\, D_2(\theta,\varphi) = C_{n,2}\, \left(K(\theta,\theta)\, K(\varphi,\varphi) - |K(\theta,\varphi)|^2\right) \end{aligned}$$

and

$$p(\theta) = C_{n,2} \int_0^{2\pi} D_2(\theta,\varphi)\, d\varphi = C_{n,2}\; 2\pi(n-1)\, D_1(\theta) = C_{n,1}\, K(\theta,\theta).$$

g) Since $K(\theta,\theta) = n$ and $\int_0^{2\pi} p(\theta)\, d\theta = 1$, we obtain that $C_{n,1} = \frac{1}{2\pi n}$, therefore

$$p(\theta) = \frac{1}{2\pi}, \quad \theta \in [0, 2\pi],$$

and $C_{n,2} = C_{n,1}/(2\pi(n-1)) = 1/(4\pi^2\, n(n-1))$, so

$$p(\theta,\varphi) = \frac{1}{4\pi^2 n(n-1)}\, \left(n^2 - |K(\theta,\varphi)|^2\right).$$

Since

$$|K(\theta,\varphi)| = \left|\sum_{l=0}^{n-1} e^{il(\theta-\varphi)}\right| = \left|\frac{e^{in(\theta-\varphi)}-1}{e^{i(\theta-\varphi)}-1}\right| = \left|\frac{\sin\left(\frac{n(\theta-\varphi)}{2}\right)}{\sin\left(\frac{\theta-\varphi}{2}\right)}\right|,$$

we finally obtain that

$$p(\theta,\varphi) = \frac{1}{4\pi^2}\, \frac{n}{n-1} \left(1 - \frac{\sin\left(\frac{n(\theta-\varphi)}{2}\right)^2}{\left(n \sin\left(\frac{\theta-\varphi}{2}\right)\right)^2}\right).$$

NB: two lines above, we have used

$$\left|e^{i\theta}-1\right| = \left|e^{i\theta/2}\, \left(e^{i\theta/2} - e^{-i\theta/2}\right)\right| = |2\, \sin(\theta/2)|\,.$$