Homework 7: Stieltjes Transform

1. Basic properties. Let μ be a probability distribution on $\mathbb{R}$ and $g_\mu : \mathbb{C}\backslash\mathbb{R} \to \mathbb{C}$ be its Stieltjes transform, defined as

$$g_\mu(z) = \int_{\mathbb{R}} \frac{1}{x-z}\, d\mu(x), \quad z \in \mathbb{C}\backslash\mathbb{R}.$$

a) Writing $z = u + iv$, decompose $g_\mu(z)$ into real and imaginary parts.

b) It can be shown that g_μ is analytic on $\mathbb{C}\backslash\mathbb{R}$ (no proof required here).

c) Show that $\mathrm{Im}\, g_\mu(z) > 0$, if $\mathrm{Im} z > 0$.

d) Show that $\lim_{v\to+\infty} v\,|g_\mu(iv)| = 1$.

e) Show that $g_\mu(\overline{z}) = \overline{g_\mu(z)}$.

NB: If a function g satisfies properties b) to e), then it is the Stieltjes transform of a distribution μ (BTW, property c) ensures that $\mu(B) \geq 0$ for all $B \in \mathcal{B}(\mathbb{R})$ and property d) ensures that $\mu(\mathbb{R}) = 1$).

2. Inversion formula. a) Let μ be any probability distribution on $\mathbb{R}$. For any $a < b$ continuity points of F_μ, prove that

$$\frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_a^b \mathrm{Im}\, g_\mu(x + i\varepsilon)\, dx = \mu(\,]a, b[\,).$$

b) Assume now that μ has a pdf p_μ. Show that for any $x \in \mathbb{R}$,

$$p_\mu(x) = \frac{1}{\pi} \lim_{\varepsilon\downarrow 0} \mathrm{Im}\, g_\mu(x + i\varepsilon).$$

3. a) Let $x_0 \in \mathbb{R}$, $y_0 \geq 0$ and $g_0(z) = \frac{1}{x_0 - iy_0 - z}$ for $\mathrm{Im} z > 0$. Deduce the distribution corresponding to the Stieltjes transform g_0 (separate the cases $y_0 = 0$ and $y_0 > 0$). What are the moments of this distribution?

b) Let g_μ be the Stieltjes transform solution of the quadratic equation

$$g_\mu(z)^2 + z\, g_\mu(z) + 1 = 0.$$

Deduce what distribution μ corresponds to g_μ.

c) Let g_μ be the Stieltjes transform solution of the quadratic equation

$$z\, g_\mu(z)^2 + z\, g_\mu(z) + 1 = 0.$$

Deduce what distribution μ corresponds to g_μ.

NB: Both the above quadratic equations have two solutions, but notice that for each of them, only one solution is a Stieltjes transform satisfying the properties of Exercise 1.

4. Limiting eigenvalue profile and distribution.
Let $\rho \in [-1, +1]$ and $T^{(n)}$ be the $n \times n$ Toeplitz matrix whose entries are given by

$$T^{(n)}_{jk} = \rho^{|j-k|}.$$

Using the Grenander-Szegö theorem, compute *both* the limiting eigenvalue profile $g(x)$ and the limiting eigenvalue distribution $p(y)$ of $T^{(n)}$ as $n \to \infty$.