Homework 6: Moments

1. Let μ be a generic distribution on $\mathbb{R}$.
a) Show that the growth of the odd moments of μ is controlled by the growth of its even moments.
b) Show that

$$|m_k| \leq C^k \quad \forall k \geq 0 \quad \Longrightarrow \quad \limsup_{k\to\infty} \frac{1}{2k}(m_{2k})^{\frac{1}{2k}} < \infty \quad \Longrightarrow \quad \sum_{k=0}^{\infty} m_{2k}^{-\frac{1}{2k}} = \infty.$$

2. Compute the moments of the following distributions
and tell which of them are uniquely determined by their moments, using Carleman's condition.

a) Let μ be the semi-circle distribution, whose pdf is given by

$$p_\mu(x) = \frac{1}{2\pi}\sqrt{4-x^2}\,1_{\{|x|\leq 2\}}.$$

Hint: use induction and the change of variable $x = 2\sin t$. Watch out that the induction is tricky!

b) Let μ be the standard real Gaussian distribution $\mathcal{N}_\mathbb{R}(0,1)$.

Hint: use induction and the following integration by parts formula, which needs to be proven also: for any continuously differentiable $f : \mathbb{R} \to \mathbb{R}$ such that there exist $C > 0$ and $k \geq 1$ with $\max\{|f(x)|, |f'(x)|\} \leq C\,(1+x^2)^k$ for all $x \in \mathbb{R}$, we have

$$\int_\mathbb{R} x\, f(x)\, d\mu(x) = \int_\mathbb{R} f'(x) d\mu(x).$$

c) Let μ be the log-normal distribution, whose pdf is given by

$$p_\mu(x) = \frac{1}{\sqrt{2\pi x}}\exp(-(\log x)^2/2), \quad x > 0.$$

Hint: If $X \sim \mathcal{N}_\mathbb{R}(0,1)$, then μ is the distribution of the random variable $Y = e^X$.

d) Let μ be the discrete distribution defined by

$$\mu(e^j) = C\,\exp(-j^2/2), \quad j \in \mathbb{Z},$$

with C an appropriate normalization constant.

e) Let $\lambda > 0$ and μ be the distribution whose pdf is given by

$$p_\mu(x) = C_\lambda\,\exp(-x^\lambda), \quad x > 0,$$

with C_λ an appropriate normalization constant. For which values of λ is the distribution μ_λ uniquely determined by its moments? (an exact computation of the moments is not required here).

Hint: use the approximation $\Gamma(x) = \int_0^\infty y^{x-1}\, e^{-y}\, dy \sim [x]!$ as $x \to \infty$.

please turn the page

3. Proof of the central limit theorem via moments

Let $(Y_j)_{j=1}^{\infty}$ be a sequence of real i.i.d. random variables such that

$$\mathbb{E}(Y_j) = 0, \quad \text{and} \quad \mathbb{E}(Y_j^2) = 1.$$

Let $X_n = \frac{1}{\sqrt{n}} \sum_{j=1}^n Y_j$ and $F_n(t) = \mathbb{P}(X_n \leq t)$, $t \in \mathbb{R}$. The central limit theorem asserts that

$$\lim_{n\to\infty} F_n(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)\, dx, \quad \forall t \in \mathbb{R}.$$

This result says that the sequence F_n converges weakly to the $\mathcal{N}_{\mathbb{R}}(0,1)$ distribution. Notice however that F_n is *not* the empirical distribution of the Y_j's; moreover, it is a deterministic distribution!

The central limit theorem is usually proved using Fourier transform. The aim of the present exercise is to prove it *using moments*, under the following additional assumptions:

a) Y_j are bounded random variables (i.e. $|Y_j(\omega)| \leq C$, $\forall \omega$).

b) $\mathbb{E}(Y_j^{2l+1}) = 0$ for all $l \geq 0$.

Hint: for the even moments, use the multinomial expansion

$$(y_1 + \ldots + y_n)^{2k} = \sum_{\substack{j_1,\ldots j_n \geq 0 \\ j_1+\ldots+j_n=2k}} \binom{2k}{j_1, \ldots, j_n} y_1^{j_1} \cdots y_n^{j_n},$$

and divide the sum into two parts as follows: $\sum_{\substack{j_1,\ldots j_n \geq 0 \\ j_1+\ldots+j_n=2k}} = \sum_{\substack{j_1,\ldots j_n \in \{0,2\} \\ j_1+\ldots+j_n=2k}} + \sum_{\substack{\exists 1\leq i\leq n\,:\, j_i \notin\{0,2\} \\ j_1+\ldots+j_n=2k}}$.