Homework 5

1. a) Matrix norms: For a given $n \times n$ matrix $A = (a_{jk})$, we define

$$\|A\|_1 = \sup_{x \in \mathbb{C}^n, x \neq 0} \frac{\|Ax\|}{\|x\|} \quad \text{and} \quad \|A\|_2 = \sqrt{\frac{1}{n}\mathrm{Tr}(A^*A)}.$$

Show that

$$\left|\frac{1}{n}\mathrm{Tr}(A)\right| \leq \|A\|_2 \leq \|A\|_1.$$

For two $n \times n$ matrices A and B, show moreover that

$$\|AB\|_1 \leq \|A\|_1 \, \|B\|_1 \quad \text{and} \quad \|AB\|_2 \leq \|A\|_1 \, \|B\|_2.$$

b) Relation with eigenvalues:
Let $\alpha_1, \ldots, \alpha_n$ be the eigenvalues of A and $\beta_1, \ldots, \beta_n$ be the eigenvalues of A^*A.
What do we know *a priori* about the β's?

In the case where A is Hermitian (i.e. $A = A^*$), what do we know *a priori* about the α's?
and what is the relation between the α's and the β's?

Show that in general, the following holds:

$$\|A\|_1 = \max_{j \in \{1,\ldots,n\}} \sqrt{\beta_j}, \quad \|A\|_2 = \sqrt{\frac{1}{n}\sum_{j=1}^n \beta_j}, \quad \text{and} \quad \frac{1}{n}\mathrm{Tr}(A^m) = \frac{1}{n}\sum_{j=1}^n \alpha_j^m, \quad \text{for any } m \geq 0.$$

c) Asymptotic equivalence of Toeplitz and ciculant matrices:
(first step in the proof of the Grenander-Szegö theorem = Lemma 1 of the class)

Let $t_0 = 2$, $t_1 = t_{-1} = -1$ and $t_l = 0$ for $|l| > 1$. Let $(T^{(n)}, n \geq 1)$ be the sequence of Toeplitz matrices built from the sequence $(t_l,\, l \in \mathbb{Z})$. For each n, let $C^{(n)}$ be the circulant matrix corresponding to $T^{(n)}$ (that is, the matrix $T^{(n)}$ with 0 replaced by -1 in the lower left and upper right corners).

Show that there exists $K > 0$ such that

$$\sup_{n \geq 1} \|T^{(n)}\|_1 \leq K \quad \text{and} \quad \sup_{n \geq 1} \|C^{(n)}\|_1 \leq K \quad \text{and} \quad \|T^{(n)} - C^{(n)}\|_2 \leq \frac{K}{\sqrt{n}}.$$

Noticing that for any $m \geq 0$, (this can be proved easily by induction)

$$(T^{(n)})^m - (C^{(n)})^m = \sum_{j=1}^m (C^{(n)})^{j-1} \, (T^{(n)} - C^{(n)}) \, (T^{(n)})^{m-j},$$

show, using most of the preceding statements, that if $\lambda_k^{(n)}$, $\mu_k^{(n)}$ are the eigenvalues of $T^{(n)}$, $C^{(n)}$, respectively, then for any fixed $m \geq 0$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n (\lambda_k^{(n)})^m = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n (\mu_k^{(n)})^m.$$

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2. Proof of Lemma 2 of the class

Let $(t_l,\, l \in \mathbb{Z})$ be a sequence of complex numbers such that $t_l = 0$ for all $|l| > l_0$ and $t_{-l} = \overline{t_l}$ for all $|l| \le l_0$. Let $T^{(n)}$ be the $n \times n$ matrix with entries $T^{(n)}_{jk} = t_{k-j}$ and g be the function defined as

$$g(x) = \sum_{l=-l_0}^{l_0} t_l \, e^{ilx}, \quad x \in [0, 2\pi].$$

Show that if $\lambda^{(n)}$ is an eigenvalue of $T^{(n)}$, then

$$\lambda^{(n)} \in \mathbb{R} \quad \text{and} \quad \inf_{x \in [0,2\pi]} g(x) \le \lambda^{(n)} \le \sup_{x \in [0,2\pi]} g(x).$$

Hint: compute $u^*(T^{(n)})u$ for the eigenvector u corresponding to $\lambda^{(n)}$ and use the fact that

$$t_l = \frac{1}{2\pi} \int_0^{2\pi} g(x) \, e^{-ilx} \, dx.$$