Homework 4: Circular Unitary Ensemble

Let U be an $n \times n$ complex unitary matrix picked according to the Haar distribution on $U(n)$. U has n complex eigenvalues $\lambda_1, ..., \lambda_n$ which are all located on the unit circle: $|\lambda_j| = 1$. We may therefore write $\lambda_j = e^{i\theta_j}$, where $\theta_j \in [0, 2\pi[$. The joint distribution of (the arguments of) the eigenvalues of U is given by

$$p(\theta_1, ..., \theta_n) = C_n \prod_{j<k} \left| e^{i\theta_k} - e^{i\theta_j} \right|^2, \quad \theta_j \in [0, 2\pi[.$$

The goal of this exercise is to show that the first order and second order marginal distributions of $p(\theta_1, ..., \theta_n)$ are given respectively by

$$p(\theta) = \frac{1}{2\pi} \quad \text{and} \quad p(\theta, \varphi) = \frac{1}{4\pi^2} \frac{n}{n-1} \left(1 - \left(\frac{\sin(n(\theta - \varphi)/2)}{n \sin((\theta - \varphi)/2)} \right)^2 \right), \quad \theta, \varphi \in [0, 2\pi[.$$

a) Following the reasoning developed in class for the Laguerre Unitary Ensemble, express $p(\theta_1, ..., \theta_n)$ in terms of the self-reproducing kernel $K(\theta, \varphi) = \sum_{l=0}^{n-1} e^{il\theta} \, \overline{e^{il\varphi}}$.

b) Show that (0) $K(\varphi, \theta) = \overline{K(\theta, \varphi)}$.

(i) $\int_0^{2\pi} K(\theta, \theta) \, d\theta = 2\pi \, n$.

(ii) $\int_0^{2\pi} K(\theta, \varphi) \, K(\varphi, \psi) \, d\varphi = 2\pi \, K(\theta, \psi)$.

c) For $m \in \{1, \ldots, n\}$, let $D_m(\theta_1, \ldots, \theta_m) = \det \left(\{K(\theta_j, \theta_k)\}_{j,k=1}^m \right)$. Prove Mehta's lemma:

$$\int_0^{2\pi} D_m(\theta_1, \ldots, \theta_m) \, d\theta_m = 2\pi \, (n - m + 1) \, D_{m-1}(\theta_1, \ldots, \theta_{m-1}).$$

Hint: recall that for a $m \times m$ matrix A and for any $j \in \{1, \ldots, m\}$,

$$\det A = \sum_{k=1}^{m} (-1)^{j+k} \, a_{jk} \, \det(A(j,k)) = \sum_{k=1}^{m} (-1)^{j+k} \, a_{kj} \, \det(A(k,j)),$$

where $A(j, k)$ is the matrix A whose row j and column k have been removed.

d) Deduce the formulas stated above for $p(\theta)$ and $p(\theta, \varphi)$.