Homework 3: Joint eigenvalue distribution of $W = HQH^*$

Let H be an $n \times n$ complex matrix with i.i.d.$\sim \mathcal{N}_{\mathbb{C}}(0,1)$ entries
and Q be an $n \times n$ deterministic and positive definite matrix.

The goal of this homework is to determine the joint distribution of the eigenvalues of the $n \times n$ matrix $W = HQH^*$.

a) Show that W is positive semi-definite.

b) Let $M = \mathrm{diag}(\mu_1, \ldots, \mu_n)$, where $\mu_1, \ldots, \mu_n$ are the (positive) eigenvalues of Q. Show that W and HMH^* have the same distribution.

c) Compute the joint distribution of the entries of $\widetilde{H} = HM^{1/2}$.

[NB: $M^{1/2} = \mathrm{diag}(\sqrt{\mu_1}, \ldots, \sqrt{\mu_n})$]

d) Compute the the joint distribution of the entries of the matrix $\widetilde{W} = \widetilde{H}^* \widetilde{H}$.

[NB: this is *not* a typo; we do not consider here $\widetilde{W} = \widetilde{H}\widetilde{H}^*$.]

e) Compute the joint distribution of the eigenvalues of $\widetilde{W}$ (which is the same as that of W: why?).

[NB: do not worry if you cannot get a completely closed form expression!]