Homework 1: General properties of eigenvalues

Let $A = (a_{jk})$ be a $n \times n$ matrix with complex-valued entries and λ be an eigenvalue of A (i.e., there exists $u \in \mathbb{C}^n$ such that $u \neq 0$ and $Au = \lambda u$). Prove the following statements:

a1) If A is Hermitian (i.e. $A^* = A$), in particular if A is real symmetric, then $\lambda \in \mathbb{R}$.

a2) If A is anti-Hermitian (i.e. $A^* = -A$), then $\lambda \in i\mathbb{R}$.

b1) If $a_{jk} \in \mathbb{R}$ for all $j, k \in \{1, \ldots, n\}$, then $\overline{\lambda}$ is also an eigenvalue of A.

b2) If $a_{jk} \in i\mathbb{R}$ for all $j, k \in \{1, \ldots, n\}$, then $-\overline{\lambda}$ is also an eigenvalue of A.

Note also these consequences:

a2)+b1) If A is real anti-symmetric, then $\lambda \in i\mathbb{R}$ and $\overline{\lambda} = -\lambda$ is also an eigenvalue of A.

a1)+b2) If A is anti-symmetric and $a_{jk} \in i\mathbb{R}$ for all $j, k \in \{1, \ldots, n\}$, then $\lambda \in \mathbb{R}$ and $-\overline{\lambda} = -\lambda$ is also an eigenvalue of A (in this case, A is actually Hermitian!).

c) If A is positive semi-definite, then $\lambda \geq 0$.

d1) If A is unitary (i.e. $AA^* = I = A^*A$), then $|\lambda| = 1$.

d2) If $a_{jk} = \frac{1}{\sqrt{n}} \exp(\frac{2\pi ijk}{n})$ for all $j, k \in \{1, \ldots, n\}$, to which set do the eigenvalues of A belong?

Hint: Compute explicitly AA^* and A^2 first!

e1) Part of Perron-Frobenius theorem: if A is non-negative (i.e. $a_{jk} \geq 0$ for all $j, k \in \{1, \ldots, n\}$), then

$$|\lambda| \leq \max_{j \in \{1,\ldots,n\}} \sum_{k=1}^{n} a_{jk}.$$

e2) If A is stochastic (i.e. $a_{jk} \geq 0$ for all $j, k \in \{1, \ldots, n\}$ and $\sum_{k=1}^{n} a_{jk} = 1$ for all $j \in \{1, \ldots, n\}$), then $|\lambda| \leq 1$.

f) Geršgorin discs' theorem: without any assumption on A, it is always true that

$$\lambda \in \bigcup_{j=1}^{n} B\left(a_{jj}, \sum_{k=1, k\neq j}^{n} |a_{jk}|\right),$$

where $B(a, r) = \{z \in \mathbb{C} : |z - a| \leq r\}$.