Statistical Ensembles of Complex, Quaternion, and Real Matrices

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Statistical ensembles of complex, quaternion, and real matrices with Gaussian probability distribution, are studied. We determine the over-all eigenvalue distribution in these three cases (in the real case, under the restriction that all eigenvalues are real). We also determine, in the complex case, all the correlation functions of the eigenvalues, as well as their limits when the order N of the matrices becomes infinite. In particular, the limit of the eigenvalue density as $N \to \infty$ is constant over the whole complex plane.

INTRODUCTION

IN order to obtain a theoretical description of highly excited regions of heavy-nuclei spectra, several authors[1-6] have developed a theory of statistical matrix ensembles. The energy levels are the eigenvalues of the Hamiltonian H, which is regarded as an Hermitian matrix of very large order N. In the absence of any precise knowledge of H, one assumes a reasonable probability distribution for its matrix elements, from which one deduces statistical properties of its spectrum. An important simplification followed the introduction of unitary instead of Hermitian matrix ensembles.[4] However, the physical origin of the problem—more precisely, the need to interpret part of the spectrum of a matrix as part of the energy spectrum of a physical system—has restricted the attention to matrix ensembles whose spectrum is contained in a one-dimensional line of the complex plane (the real axis for Hermitian, the unit circle for unitary matrices). In the present paper we study the eigenvalue distributions of matrix ensembles for which any point of the complex plane may belong to the spectrum. Apart from the intrinsic interest of the problem,[6] one may hope that the methods and results will provide further insight in the cases of physical interest or suggest as yet lacking applications.

The definition of matrix ensembles consists of two parts:

(1) The definition of an algebraic set of matrices Z. We shall consider: C, complex $N \times N$ matrices; Q, quaternion $N \times N$ matrices; R, real $N \times N$ matrices. The order is that of increasing difficulty. These are the Z's of Dyson's classification (Ref. 5, especially Theorem 7) and they can be characterized as the only irreducible matrix algebras over the real field [Refs. 5, 7 (especially Chap. 3)].

(2) The choice of a measure or probability distribution on Z. Z is a finite-dimensional vector space over the real field. The linear measure $d\mu_L$ which suggests itself naturally, is not suited for a probabilistic interpretation, for the measure of the whole space is infinite. We choose instead

$$d\mu(S) = d\mu_L(S) \exp\left[-(1/4a^2) \operatorname{Tr} S^\dagger S\right] \tag{0.1}$$

(S generic element of Z, a = real positive constant) which satisfies the two properties:

(i) It is invariant under the adjoint representation of Z_U, where Z_U is the group of unitary matrices in Z. More precisely, for any $U \in Z_U$,

$$d\mu(S) = d\mu(U^\dagger S U).$$

(ii) The matrix elements of S are statistically independent.

These two properties are likely to determine $d\mu(S)$ uniquely.[2] We study the three matrix ensembles thus obtained Z_C, Z_Q, and Z_R in Secs. 1, 2, and 3, respectively, with decreasing success. We obtain for Z_C the over-all eigenvalue distribution and all the correlation functions of n eigenvalues ($1 \le n \le N$); for Z_Q the over-all eigenvalue distribution only; for Z_R the over-all eigenvalue distribution in the restrictive case where all eigenvalues are real.

1. COMPLEX MATRICES

Z is the algebra of complex $N \times N$ matrices $S = (S_{ij})$. The linear measure is defined by

$$d\mu_L(S) = \prod_{i,j} dS_{ij}\, dS_{ij}^*, \tag{1.1}$$

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where $dz\,dz^*$ means $2\,dx\,dy$ if $z = x + iy$, $z^* = x - iy$. $d\mu(S)$ is defined by $(0 - 1)$.

From now on, we take $4a^2 = 1$. We define the eigenvalue distribution $P_N(z_1, \cdots, z_N)$ by

$$\int\!\!\!\!- d\mu(S) = P_N(z_1, \cdots, z_N) \prod_{i=1}^{N} dz_i\, dz_i^*, \qquad (1.2)$$

where z_i $(i = 1, \cdots, N)$ are the eigenvalues of the generic element $S \in Z$.

$\int\!\!\!\!-$ means: for fixed (z_i), integration over all other variables. We now compute P_N. Any $S \in Z$ with distinct eigenvalues (we forget matrices with two equal eigenvalues, which form a set of measure 0) can be diagonalized

$$S = XAX^{-1}, \qquad (1.3)$$

where A is diagonal with $A_{ii} = z_i$, and X is regular and is defined modulo multiplication on the right by any element of the commutator $\mathcal{A}'$ of A. $\mathcal{A}'$ consists of all complex diagonal matrices. The correspondence $S \to (A, X \bmod \mathcal{A}')$ is $1 \to N!$ because there are $N!$ ways to order the eigenvalues.

Infinitesimal variations dA, dX of A, X produce a variation dS of S

$$dS = X(dA + [dR, A])X^{-1}, \qquad (1.4)$$

where $dR = X^{-1}dX$. From (1.4) and the invariance property: $d\mu_{\mathrm{L}}(S) = d\mu_{\mathrm{L}}(XSX^{-1})$ for any regular $X \in Z$, we get

$$d\mu_{\mathrm{L}}(S) = \prod_i dz_i\, dz_i^* \prod_{i\neq j} [dR, A]_{ij}[dR, A]_{ij}^*, \qquad (1.5)$$

$$d\mu_{\mathrm{L}}(S) = \prod_i dz_i\, dz_i^* \prod_{i<j} |z_i - z_j|^4 \prod_{i\neq j} dR_{ij}\, dR_{ij}^*. \qquad (1.6)$$

We substitute (1.6) into (1.2) and get

$$P_N(z_1, \cdots, z_N) = \frac{1}{N!} \prod_{i<j} |z_i - z_j|^4\, J, \qquad (1.7)$$

where

$$J = \int \prod_{i\neq j} dR_{ij}\, dR_{ij}^* \exp\,[-\mathrm{Tr}\, S^\dagger S]. \qquad (1.8)$$

$d\mu_0(X) = \prod_{i,j} dR_{ij}\, dR_{ij}^*$ is the invariant measure on the group $Gl(N, C)$ which we note simply $\mathcal{G}$ and $\prod_{i\neq j} dR_{ij}\, dR_{ij}^*$ is the quotient measure on $\mathcal{G}/\mathcal{A}'$ (which is *not* a group). We need the following

Lemma: Any $X \in \mathcal{G}$ can be written in one and only one way as

$$X = UYV, \qquad (1.9)$$

where U is unitary $(U \in Z_{\mathrm{U}})$, Y is triangular $(Y_{ij} = 0$ for $i > j)$, and $Y_{ii} = 1$ $(i = 1, \cdots, N)$. The set of the matrices with these two properties is a unimodular subgroup $\mathcal{Y}$ of $\mathcal{G}$. V is diagonal with real positive elements. The set of these matrices is also a subgroup $\mathcal{V}$ of $\mathcal{G}$.

(In other words, we have decomposed $\mathcal{G}$ as the product of three subgroups Z_{U}, $\mathcal{Y}$, and $\mathcal{V}$. None of them is invariant and this decomposition has no simple algebraic property. However, it will provide a useful decomposition of $d\mu_0(X)$.

Proof: Any set X^α $(\alpha = 1, \cdots, N)$ of N linearly independent vectors can be brought by a unitary change of basis into a form where the components X_j^α satisfy $X_j^\alpha = 0$ for $j > \alpha$. We apply this to the column vectors of X and obtain $X = U_1Y_1$ with U_1 unitary and Y_1 triangular. Y_1 can then be written as $Y_1 = U_0YV$ with Y and V as above and U_0 diagonal and unitary. Therefore $X = UYV$ with $U = U_1U_0$. If now $X = UYV = U'Y'V'$, then $B = U'^{-1}U = Y'V'V^{-1}Y^{-1}$ is unitary (left-hand side) and triangular with real positive diagonal elements (right-hand side); therefore $B = 1$, and the decomposition is unique.

We now change the variables in $d\mu_0(X)$ from dR to dU, dY, dV. We start from

$$dR = X^{-1}\, dX = V^{-1}\, dV + V^{-1}Y^{-1}\, dY\, V + V^{-1}Y^{-1}U^{-1}\, dU\, YV. \qquad (1.10)$$

The first term contributes to the real part of the diagonal elements; The second term, to the dR_{ij} with $i < j$. In the third term, $dL = -iU^{-1}dU$ is Hermitian. From $Y \in \mathcal{Y}$, $Y^{-1} \in \mathcal{Y}$ it follows that, for $i \geq j$,

$$(Y^{-1}\, dL\, Y)_{ij} = dL_{ij} + (\text{linear combination of } dL_{kl} \text{ with } k - l > i - j). \qquad (1.11)$$

It follows from these remarks that

$$d\mu_0(X) = d\mu_0(U)\, d\mu_0(Y)\, d\mu_0(V), \qquad (1.12)$$

where

$$d\mu_0(U) = \prod_{i>j} dL_{ij}\, dL_{ij}^* \prod_i dL_{ii} = \prod_{i,j} dL_{ij}, \qquad (1.13)$$

$$d\mu_0(Y) = \prod_{i<j} (Y^{-1}\, dY)_{ij}(Y^{-1}\, dY)_{ij}^*, \qquad (1.14)$$

$$d\mu_0(V) = \prod_i (2V_{ii}^{-1}\, dV_{ii}), \qquad (1.15)$$

are the invariant measures on Z_{U}, $\mathcal{Y}$, and $\mathcal{V}$ respectively. We need the quotient measure on $\mathcal{G}/\mathcal{A}'$,

$$\prod_{i\neq j} dR_{ij}\, dR_{ij}^* = d\mu_0(\mathcal{G})/d\mu_0(\mathcal{A}'). \qquad (1.16)$$

$\mathcal{A}'$ is the direct product of $\mathcal{V}$ and the group $\mathcal{U}_0$ of unitary diagonal matrices, with

$$d\mu_0(\mathcal{A}') = d\mu_0(\mathcal{U}_0)\, d\mu_0(\mathcal{V}). \qquad (1.17)$$

We now come back to (1.8). From (1.3)

$$\operatorname{Tr} S^\dagger S = \operatorname{Tr} A^\dagger H A H^{-1}, \tag{1.18}$$

where H should be $X^\dagger X$. But V commutes with A and therefore disappears, as expected, which enables us to take $H = Y^\dagger Y$. We then get from (1.8), (1.12), (1.16)

$$J \int d\mu_0(\mathfrak{U}_0)$$
$$= \int \exp\left[-\operatorname{Tr} S^\dagger S\right] d\mu_0(U)\, d\mu_0(Y) \tag{1.19}$$

or

$$J = \frac{\Omega_U}{(2\pi)^N} \int \exp\left[-\operatorname{Tr} S^\dagger S\right] d\mu_0(Y), \tag{1.20}$$

where $(2\pi)^N$ is the volume of $\mathfrak{U}_0$ and $\Omega_U = \int d\mu_0(U)$ is the volume of Z_U. We next perform a last change of variables from Y to H. Any $n \times n$ upper left block Y_n of any $Y \in \mathcal{Y}$ has determinant 1. This implies that the jacobian of the transformation from $(dY)_{ij}$ to $(Y^{-1}dY)_{ij}$ is 1. Therefore

$$d\mu_0(Y) = \prod_{i<j} dY_{ij}\, dY_{ij}^*. \tag{1.21}$$

H is defined by $H = Y^\dagger Y$ or more explicitly

$$\begin{cases} H_{ij} = Y_{ij} + \sum_{k<i} Y_{ki}^* Y_{kj} & \text{for} \quad i < j, \\ H_{ij} = Y_{ji}^* + \sum_{k<j} Y_{ki}^* Y_{kj} & \text{for} \quad i > j. \end{cases} \tag{1.22}$$

Each $n \times n$ upper left block H_n of H can be defined by $H_n = Y_n^\dagger Y_n$. Therefore $\det Y_n = 1$ determines the diagonal elements of H_n by the condition $\det H_n = 1$. Each diagonal element of H is then defined (by induction) as a polynomial with integer coefficients in the nondiagonal elements.

It follows immediately from (1.22) that the correspondence $Y \to H$ is one to one and that

$$d\mu_0(Y) = \prod_{i<j} dH_{ij}\, dH_{ij}^* = \prod_{i\neq j} dH_{ij}. \tag{1.23}$$

We now perform the integration over H in N steps; each one consists of integrating over the variables of the last row and column of H, and brings back to the original problem for matrices of order smaller by one unit, obtained from the original ones by removing the last row and column. The proof rests on a recursion formula which we now derive:

Let H', A', etc. be the relevant matrices of order n; H, A the matrices of order $n - 1$, more precisely the $(n-1) \times (n-1)$ upper left blocks of H', A'. Greek (Latin) indices run from 1 to n $(n-1)$. From $\det H' = 1$, it follows that $H'^{-1}_{\beta\alpha} = \Delta'_{\alpha\beta}$, where $\Delta'_{\alpha\beta}$ is the minor (α, β) of H'. Similarly $H^{-1}_{ji} = \Delta_{ij}$, where Δ_{ij} is the minor (ij) of H. Now let

$$\phi_n = \operatorname{Tr} S'^\dagger S' = \operatorname{Tr} A'^\dagger H' A' H'^{-1}$$
$$= \sum_{\alpha\beta} H'_{\alpha\beta} z_\alpha^* z_\beta \Delta'_{\alpha\beta}. \tag{1.24}$$

We separate the last row and column; let $e = (e_i)$, $i = 1, \cdots, n-1$, where $e_i = H'_{in}$. Then

$$\phi_n = |z_n|^2 H'_{nn} - z_n \sum_{i,k} e_i z_i^* e_k^* \Delta_{ik} - z_n^* \sum_{j,k} e_j^* z_j e_k \Delta_{kj}$$
$$+ \sum_{i,j,k,l} H_{ij} z_i^* z_j (H'_{nn}\Delta_{ij} - e_l^* e_k \Delta_{ij}^{kl}), \tag{1.25}$$

where Δ_{ij}^{kl} is the minor of H obtained by removing the rows i, k and the columns j, l. From $\det H' = \det H = 1$, we get

$$H'_{nn} = 1 + \sum_{k,l} e_l^* e_k \Delta_{kl}. \tag{1.26}$$

Substituting in (1.26) and using the elementary identity

$$\Delta_{ij}^{kl} = \Delta_{ij}\Delta_{kl} - \Delta_{il}\Delta_{kj}, \tag{1.27}$$

we get, after straightforward algebra,

$$\phi_n = |z_n|^2 + \phi_{n-1}$$
$$+ \langle e^* |\, H^{-1}(A^\dagger - z_n^*) H (A - z_n) H^{-1} \,| e\rangle, \tag{1.28}$$

where

$$\langle e^* |\, B \,| e\rangle = \sum_{k,l} e_l^* B_{lk} e_k. \tag{1.29}$$

We now substitute (1.28) for $n = N$ into (1.20) and get

$$J = \frac{\Omega_U}{(2\pi)^N} \int \exp\left[-|z_N|^2 - \phi_{N-1}\right]$$
$$\times \prod_{i<j\leq N-1} dH_{ij}\, dH_{ij}^* \int \prod_{i=1}^{N-1} de_i\, de_i^*$$
$$\times \exp\left[-\langle e^* |\, H^{-1}(A^\dagger - z_N^*) H (A - z_N) H^{-1} \,| e\rangle\right]. \tag{1.30}$$

The last integration is straightforward and gives

$$(2\pi)^{N-1}\left[\prod_1^{N-1} |z_i - z_N|^2\right]^{-1}.$$

The same procedure can be repeated N times and gives

$$J = \Omega_U (2\pi)^{N(N-3)/2} \left[\prod_{i<j} |z_i - z_j|^2\right]^{-1}$$
$$\times \exp\left[-\sum_{i=1}^N |z_i|^2\right]. \tag{1.31}$$

Ω_U is easily computed.[4] With the normalization

(1.13), we get

$$\Omega_U = (2\pi)^{N(N+1)/2}/1!2!\cdots(N-1)!. \qquad (1.32)$$

(1.31), (1.32), and (1.7) then give

$$P_N(z_1,\cdots,z_N) = \frac{(2\pi)^{N(N-1)}}{1!2!\cdots N!}\prod_{i<j}|z_i - z_j|^2 \times \exp\left[-\sum |z_i|^2\right]. \qquad (1.33)$$

P_N is normalized according to

$$\int P_N(z_1,\cdots,z_N)\prod_i dz_i\,dz_i^* = \int d\mu(S) = (2\pi)^{N^2}, \qquad (1.34)$$

which follows from (0.1) and (1.2) by direct computation. This suggests to define

$$\hat{P}_N(z_1,\cdots,z_N) = [1!2!\cdots N!(2\pi)^N]^{-1}\prod_{i<j}|z_i - z_j|^2 \times \exp\left[-\sum |z_i|^2\right] \qquad (1.35)$$

with the normalization

$$\int \hat{P}_N(z_1,\cdots,z_N)\prod dz_i\,dz_i^* = 1. \qquad (1.36)$$

We next determine the n-eigenvalue correlation functions which are defined (Ref. 4) by

$$\rho_N(z_1,\cdots,z_n) = \frac{N!}{(N-n)!}\int \hat{P}_N(z_1,\cdots,z_N)\prod_{i=n+1}^{N} dz_i\,dz_i^*, \qquad (1.37)$$

and normalized according to

$$\int \rho_N(z_1,\cdots,z_n)\prod_{i=1}^{n} dz_i\,dz_i^* = \frac{N!}{(N-n)!}. \qquad (1.38)$$

Now

$$\prod_{i<j}(z_i - z_j) = \det(z_i^{j-1}) = \sum_{\Pi}(-)^{\Pi} z_1^{p_1-1}\cdots z_N^{p_N-1} \qquad (1.39)$$

where Π is the permutation $(1,\cdots,N)\to(p_1,\cdots,p_N)$. Each term of the expansion of the right-hand side is multiplied, in P_N, by a similar term in z^*, and integrated over $N-n$ variables $z_i = r_i e^{i\theta_i}$ in $\rho_N(z_1,\cdots,z_n)$. The angular integration gives zero except if every z_i over which one integrates occurs with the same power p_i' as the complex conjugate z_i^*. The subsequent integration on r_i then gives

$$\int_0^\infty d(r^2)(r_i^2)^{p'_i}\exp(-r_i^2) = p_i'!.$$

From this remark and from (1.35), (1.37), $\rho_N(z_1,\cdots,z_N)$ is obtained as follows:

Choose a set of $N-n$ exponents (p_i'), i.e., $N-n$ integers between 0 and $N-1$, which are to be distributed among the integration variables $z_{n+1},\cdots,z_N$.

Distribute them among these variables, which gives a factor $(N-n)!$ and cancels the corresponding factor in (1.37).

Integrate over the $N-n$ angles θ_i ($i = n+1,\cdots,N$), which gives a factor $(2\pi)^{N-n}$.

Integrate over the $N-n$ variables $|z_i| = r_i$, which gives one factor $p_i'!$ for each p_i' and cancels the corresponding factor in $\hat{P}_N$.

Distribute the remaining set of exponents (p_i) [i.e., the n integers between 0 and $N-1$ not taken as (p_i')] among $z_1,\cdots,z_n$ and $z_1^*,\cdots,z_n^*$ separately.

Sum over all possible such distributions, after multiplying by the two sign factors which come from the expansion of the original determinants.

Sum over all possible partitions of $(0,\cdots,N-1)$ into the two families (p_i) and (p_i').

Note that $N!$ in the definition of ρ_N cancels $1/N!$ in the definition of $\hat{P}_N$. We then get

$$\rho_N(z_1,\cdots,z_n) = \frac{1}{(2\pi)^n}\exp\left[-\sum_{i=1}^{n}|z_i|^2\right]\det(D_{ij}), \qquad (1.40)$$

where

$$D_{ij} = \sum_{p=0}^{N-1}\frac{(z_i z_j^*)^p}{p!} \quad (i, j = 1,\cdots,n). \qquad (1.41)$$

When $N\to\infty$, the correlation functions tend to well-defined limits

$$\rho(z_1,\cdots,z_n) = \frac{1}{(2\pi)^n} \times \exp\left[-\sum_1^n |z_i|^2\right]\det[\exp(z_i z_j^*)]. \qquad (1.42)$$

We now consider in some detail the eigenvalue density in the complex plane

$$\rho_N(z) = \frac{1}{2\pi}\exp\left[-|z|^2\right]\sum_{p=0}^{N-1}\frac{|z|^{2p}}{p!}. \qquad (1.43)$$

Its limit for $N\to\infty$ is a constant: $\rho(z) = \rho = 1/2\pi$. $\rho_N(z)$ is normalized according to $\int \rho_N(z)\,dz\,dz^* = N$. One verifies directly that each term in the last sum in (1.43) contributes 1 to this integral. $\rho_N(z)$ is invariant by rotation around the origin, as was obvious from the symmetry of the problem. Let $r = |z|$. The last sum in (1.43) is the beginning of the expansion of $\exp r^2$ in powers of r^2. Therefore, for $r^2 \ll N$, $\rho_N(z)\simeq\rho$ and for $r^2 \gg N$, $\rho_N(z)\simeq 0$. More precisely, elementary bounds on the exponential series give

$$1 - 2\pi\rho_N(z) \le \exp(-r^2)\frac{r^{2N}}{N!}\frac{N+1}{N+1-r^2} \quad \text{for} \quad r^2 \le N, \qquad (1.44)$$

$$2\pi\rho_N(z) \le \exp(-r^2)\frac{r^{2N}}{N!}\frac{N}{r^2+1-N} \quad \text{for} \quad r^2 \ge N. \qquad (1.45)$$

For $r = N^{\frac{1}{2}} \pm u$, $0 \le u \lesssim 1 \ll N$, the leading term in the right-hand side of (1.44) and (1.45) is $\exp(-2u^2)/2u(2\pi)^{\frac{1}{2}}$. This gives a sharp fall of $\rho_N(z)$ from ρ to 0 when r varies in an interval of order 1 around $N^{\frac{1}{2}}$. As a consequence, the number of eigenvalues in the "tail" of the distribution, defined as

$$\delta N = \int_{|z|>N^{\frac{1}{2}}} \rho_N(z)\, dz\, dz^* \qquad (1.46)$$

is proportional to $N^{\frac{1}{2}}$. More precisely $\delta N \simeq (N/2\pi)^{\frac{1}{2}}$ for $N \gg 1$.

Electrostatic Analogy

The electrostatic model introduced by Wigner[8] and Dyson[4] can be extended to the present case. Consider N unit charges in a two-dimensional space, which is taken as the complex plane of the variable z. The positions of the charges are $z_1, \cdots, z_N$. Suppose that the charges move in an harmonic oscillator potential $\frac{1}{2}|z|^2$ centered at the origin. Then the potential energy of the system is

$$U(z_1, \cdots, z_N) = -\sum_{i<j} \log |z_i - z_j| + \frac{1}{2}\sum_i |z_i|^2. \qquad (1.47)$$

The probability distribution of the positions $z_1, \cdots, z_N$ when this Coulomb gas is in thermodynamical equilibrium at the temperature T is proportional to $\exp[-\beta U(z_1, \cdots, z_N)]$ (where $\beta = 1/kT$). For $\beta = 2$ this is proportional to P_N.

Therefore the distribution of eigenvalues of a random matrix $S \in Z_C$ is identical with the distribution of the positions of charges of a two-dimensional Coulomb gas in an harmonic oscillator potential, at a temperature corresponding to $\beta = 2$.

2. QUATERNION MATRICES

Z is now the algebra Z_Q of $N \times N$ Q matrices, i.e., matrices with coefficients in the quaternion field Q. Q can be represented as a two-dimensional complex vector space.[9] This representation associates to any $N \times N$ Q matrix a $2N \times 2N$ complex matrix. The image of matrices in Z_Q are characterized by

$$\zeta T = T^* \zeta, \qquad (2.1)$$

where * means complex conjugate and ζ is a direct sum of N 2×2 blocks of the form $\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$. (These matrices are quaternion real in Dyson's notations.[4]) The group Z_U of unitary matrices in Z is characterized by (2.1) and $T^\dagger T = 1$, and satisfies therefore also $T^T\zeta T = \zeta$, where ^{T} means transposed. Z_U is the symptectic group $Sp(N)$. Let $S \in Z_Q$. We first look for eigenvalues of S in Q, i.e., for $\lambda \in Q$ and vectors $v \in Q^N$ such that $Sv = v\lambda$. (The product of a vector $\in Q^N$ by a scalar λ is the product by λ on the right.[9]) Now,

Lemma: If $\lambda \in Q$ is an eigenvalue of S, then $\mu^{-1}\lambda\mu$ is also an eigenvalue of S for any $\mu \neq 0$, $\mu \in Q$. In fact,

$$Sv = v\lambda \quad \text{implies} \quad S(v\mu) = v\mu(\mu^{-1}\lambda\mu). \qquad (2.2)$$

Therefore the eigenvalues of S in Q constitute orbits, each of which is the set of quaternions obtained from one $\lambda \in Q$ by the internal automorphisms $\lambda \to \mu^{-1}\lambda\mu$ (which are simply three-dimensional rotations of the imaginary part). Consider now a subspace of Q isomorphic to **C**. This subspace intersects every orbit in two points which are complex conjugate in **C** (this means simply that in the three-dimensional space of purely imaginary quaterernions, we consider the intersection of a sphere with one of its diameters), and these two points determine the orbit completely. Therefore all possible information on the eigenvalues of a Q matrix in Q can be obtained from the eigenvalues in a subspace **C** of Q.

From now on we use only the $2N \times 2N$ complex representation of the matrices of Z_Q, and study their (complex) eigenvalues in the ordinary sense. One verifies directly the above mentioned result, namely that due to (2.1), the eigenvalues of any $S \in Z_Q$ are 2×2 complex conjugate. For if $Sv = zv$, $z \in \mathbf{C}$, $v \in \mathbf{C}^{2N}$, then $\zeta Sv = \zeta zv$, or $S(\zeta v^*) = z^*(\zeta v^*)$. Moreover, the eigenvector associated with z^* can be taken to be ζv^*. We shall need the following

Lemma: If $S \in Z_Q$, and if all eigenvalues of S are distinct, then $S = XAX^{-1}$, where $X, A \in Z_Q$, and A is diagonal.

In fact, S can be diagonalized as a complex matrix. One can take A as a direct sum of 2×2

[8] E. P. Wigner, Proc. 4th Can. Math. Cong., Toronto, 1959, p. 174.

[9] C. Chevalley, *Lie Groups* (Princeton University Press, Princeton, New Jersey, 1946).

blocks

$$\begin{bmatrix} z_i & 0 \\ 0 & z_i^* \end{bmatrix}, \qquad (i = 1, \cdots, N).$$

Therefore $A \in Z_Q$. The column vectors of X are the eigenvectors X^α of S. One can take $X^{2n} = \zeta(X^*)^{2n-1}$ $(n = 1, \cdots, N)$, which is equivalent to $X \in Z_Q$.

We now define measures on Z_Q. The linear measure is

$$d\mu_L(S) = \prod_{i,j} dS_{ij}. \tag{2.3}$$

[Note that because of (2.1), the factors in (2.3) are 2×2 complex conjugate.]

$$d\mu(S) = d\mu_L(S) \exp\left[-\tfrac{1}{2} \operatorname{Tr} S^\dagger S\right]. \tag{2.4}$$

The coefficient $\frac{1}{2}$ is intended to achieve greater similarity with the previous case and to compensate for the artificial doubling of the dimension of the matrices.

P_N is defined by (1.2). The first step in its calculation is the same as previously and leads to

$$d\mu_L(S) = \prod_{i=1}^{N} |z_i - z_i^*|^2 \prod_{i<j} |z_i - z_j|^4 \, |z_i - z_j^*|^4 \times \prod_{i=1}^{N} dz_i \, dz_i^* \prod_{i \neq j} dR_{ij} \tag{2.5}$$

with $dR = X^{-1}dX$, and therefore to

$$P_N(z_1, \cdots, z_N) = \left(\frac{1}{N!2^N}\right) \prod_{i=1}^{N} |z_i - z_i^*|^2 \times \prod_{i<j} |z_i - z_j|^4 \, |z_i - z_j^*|^4 J, \tag{2.6}$$

where now

$$J = \int \prod_{i \neq j} dR_{ij} \exp\left[-\tfrac{1}{2} \operatorname{Tr} S^\dagger S\right]. \tag{2.7}$$

The factor 2^N in (2.6) comes from the exchanges $z_i \leftrightarrow z_i^*$. $d\mu_0(X) = \prod_{i,j} dR_{ij}$ is the invariant measure on the group $\mathcal{G}$ of regular elements in Z_Q and $\prod_{i \neq j} dR_{ij}$ is the quotient measure on $\mathcal{G}/\mathcal{Q}'$ where $\mathcal{Q}'$ is the commutator of A in Z_Q. Any $X \in \mathcal{G}$ can be decomposed as previously as a product $X = UYV$ where U is unitary, Y is triangular ($Y_{ij} = 0$ for $i > j$) and satisfies $Y_{ii} = 1$, and V is diagonal with real positive elements. Moreover, it follows from $\zeta X = X^* \zeta$ that

$$B = Y^* V \zeta V^{-1} Y^{-1} = U^{*-1} \zeta U \tag{2.8}$$

is unitary and antisymmetric (right-hand side), and has nonzero elements only in 2×2 blocks along the diagonal (comparison of both sides). From this and the condition $V_{ii} > 0$, it follows easily that $B = \zeta$. Therefore $U \in Z_U(\subset Z_Q)$. It then follows that Y and V belong to Z_Q. In particular,

$$V_{2i-1,2i-1} = V_{2i,2i}$$

and

$$Y_{2i,2i-1} = Y_{2i-1,2i} = 0 \quad (i = 1, \cdots, N).$$

The same argument as previously then leads to

$$J = \frac{\Omega_S}{(2\pi)^N} \int \exp\left[-\frac{1}{2} \operatorname{Tr} S^\dagger S\right] d\mu_0(Y), \tag{2.9}$$

where Ω_S is the volume of the symplectic group and is defined by

$$\Omega_S = \int_{Z_U} d\mu_0(U) = \int \prod_{i \leq j} (iU^T \zeta \, dU)_{ij} \tag{2.10}$$

and

$$d\mu_0(Y) = \prod (Y^{-1} \, dY)_{ij} = \prod dY_{ij}, \tag{2.11}$$

where the product extends over the elements $i < j$, $(i, j) \neq (2k - 1, 2k)$ for $k = 1, \cdots, N$. We next change the integration variables from Y to $H = Y^\dagger Y$. Then

$$d\mu_0(Y) = \prod dH_{ij}, \tag{2.12}$$

where $\prod$ has the same meaning as in (2.11). We integrate over H in N steps as previously. We first derive a recursion formula analogous to (1.28). H', A', etc. now denote matrices of order $2n$, Δ' are the minors of H'. H, A are the $2(n-1) \times 2(n-1)$ upper left blocks of H', A'. Δ are the minors of H. From $H' \in Z_Q$ and $H' = Y'^\dagger Y'$, it follows that the 2×2 diagonal blocks of H' have the form

$$\begin{bmatrix} h_i & 0 \\ 0 & h_i \end{bmatrix}.$$

In particular,

$$H'_{2n-1,2n-1} = H'_{2n,2n} = h_n.$$

Let

$$e = (e_i), \quad e' = (e'_i) \quad [i = 1, \cdots 2(n-1)]$$

where $e_i = H'_{i,2n-1}$ and $e'_i = H'_{i,2n}$. Then

$$e' = \zeta e^*. \tag{2.13}$$

The diagonal elements of H' are defined by the condition that all upper left blocks of H' have determinant one. In particular,

$$h_n = 1 + \sum_{i,j} e_j^* e_i \Delta_{ij}, \tag{2.14}$$

$$h_n = 1 + \sum_{k,l} e_l^{*\prime} e'_k \Big(h_n \Delta_{kl} - \sum_{i,j} e_j^* e_i \Delta_{ij}^{kl}\Big). \tag{2.15}$$

(2.14) defines h_n and (2.14), (2.15) give an identity between e, e', and Δ.

$$\sum_{k,l} (e_l^* e_k - e_l^{*\prime} e_k')\Delta_{kl} = \sum_{i,j,k,l} e_l^{*\prime} e_k' e_j^* e_i (\Delta_{ij}\Delta_{kl} - \Delta_{ij}^{kl}). \quad (2.16)$$

Now from (2.13) and $H^{-1} \in Z_Q$, it follows easily that

$$\langle e^{*\prime}| H^{-1} |e'\rangle = \langle e^*| H^{-1} |e\rangle, \quad (2.17)$$

$$\langle e^{*\prime}| H^{-1} |e\rangle = -(\langle e^*| H^{-1} |e'\rangle)^*. \quad (2.18)$$

The notation is that of (1.29). From (2.17), (2.18), and $H_{ji}^{-1} = \Delta_{ij}$, it follows that both sides of (2.16) are zero, and (2.16) reduces to

$$\langle e^{*\prime}| H^{-1} |e\rangle = \langle e^*| H^{-1} |e'\rangle = 0. \quad (2.19)$$

Let now

$$\phi_n = \operatorname{Tr} A'^{\dagger} H' A' H'^{-1}.$$

We separate the last two rows and columns of the various matrices by applying (1.28) twice.

$$\phi_n = \phi_{n-1} + 2 |z_n|^2 + \langle e^*| U |e\rangle + \langle e^{*\prime}| V |e'\rangle, \quad (2.20)$$

where

$$U = H^{-1}(A^\dagger - z_n^*)H(A - z_n)H^{-1} \quad (2.21)$$

and

$$\begin{aligned} V_{ik} = &\sum [|z_n|^2 - z_n z_l' - z_n^* z_k'^*][h_n \Delta_{kl} - e_j^* e_i \Delta_{ij}^{kl}] \\ &- \sum z_i'^* z_j' H_{ij} [h_n \Delta_{ij}^{kl} - \overset{ef}{\Delta}{}_{ij}^{kl} e_f^* e_e \\ &- (\Delta_{ij} h_n - \Delta_{ij}^{ef} e_f^* e_e)\Delta_{kl}] \\ &+ (z_i'^* z_n + z_n^* z_j') e_j^* e_i (\Delta_{ij}^{kl} - \Delta_{ij}\Delta_{kl}) \end{aligned} \quad (2.22)$$

(summation over all indices e, f, i, j).

In (2.22) some terms have cancelled out because of (2.16), (2.19), and the z_i' $[i = 1, \cdots, 2(n-1)]$ are the $(z_1, z_1^*, z_2, z_2^*, \cdots, z_{n-1}, z_{n-1}^*)$ in that order. We next substitute (2.14) into (2.22). After some straightforward algebra involving repeated use of (2.13), (2.17), (2.19), (1.27) and its analogue for

$$\overset{ef}{\Delta}{}_{ij}^{kl},$$

the terms of fourth order in e, e' cancel out and we get

$$\phi_n = \phi_{n-1} + 2 |z_n|^2 + 2\langle e^*| U |e\rangle. \quad (2.23)$$

The integration over e runs exactly as in the complex case and gives after N successive steps

$$J = \Omega_S (2\pi)^{N(N-2)} [\prod_{i<j\le N} |z_i - z_j|^2 |z_i - z_j^*|^2]^{-1} \times \exp\left[-\sum_{i=1}^N |z_i|^2\right]. \quad (2.24)$$

Ω_s is easily computed.[4] With the normalization (2.10), we get

$$\Omega_S = (2\pi)^{N(N+1)}/1!3! \cdots (2N-1)!. \quad (2.25)$$

(2.6), (2.24), and (2.25) give

$$P_N(z_1, \cdots, z_N) = [(2\pi)^{N(2N-1)}/2^N N!1! \cdots (2N-1)!] \times \exp\left[-\sum_1^N |z_i|^2\right] \times \prod_{i=1}^N |z_i - z_i^*|^2 \prod_{i<j\le N} |z_i - z_j|^2 |z_i - z_j^*|^2. \quad (2.26)$$

P_N is normalized according to

$$\int P_N(z_1, \cdots, z_N) \prod dz_i \, dz_i^* = (2\pi)^{2N^2}, \quad (2.27)$$

which follows from (2.4) and (1.2) by direct computation. By analogy with (1.35), we define

$$\hat{P}_N(z_1, \cdots, z_N) = [(4\pi)^N N!1!3! \cdots (2N-1)!]^{-1} \times \exp\left[-\sum_1^N |z_i|^2\right] \times \prod_{i=1}^N |z_i - z_i^*|^2 \prod_{i<j\le N} |z_i - z_j|^2 |z_i - z_j^*|^2, \quad (2.28)$$

which is normalized according to (1.36).

The determination of the correlation functions appears to be considerably more difficult than in the complex case, and the electrostatic interpretation of P_N breaks down.

3. REAL MATRICES

Z is now the algebra of real $N \times N$ matrices $S = (S_{ij})$. The linear measure is

$$d\mu_L(S) = \prod_{i,j} dS_{ij}. \quad (3.1)$$

The eigenvalues of S now consist of ν complex-conjugate pairs $(0 \le 2\nu \le N)$ and $q = N - 2\nu$ real numbers: $z_{2i} = Z_{2i-1}^*$ for $i = 1, \cdots, \nu$ and z_i real for $i > 2\nu$.

We define $P_N^\nu(z_1, \cdots, z_N)$ by

$$⨍ d\mu(S) = P_N^\nu(z_1, \cdots, z_N) \prod_{i=1}^N dz_i \quad (3.2)$$

where ⨍ has the same meaning as in (1.2).

Any S with distinct eigenvalues can be diagonalized: $S = XAX^{-1}$. A is diagonal with $A_{ii} = z_i$. X is regular. Its first 2ν column vectors X^α are 2×2 complex conjugate and the last q are real:

$$X^{2j} = (X^{2j-1})^* \quad \text{for} \quad 1 \le j \le \nu$$

and

$$X^j = X^{j*} \quad \text{for} \quad j > 2\nu. \quad (3.3)$$

X is defined modulo multiplication on the right by any element of the group $\mathcal{W}$ of diagonal matrices W which satisfy:

$$W_{2j,2j} = W^*_{2j-1,2j-1} \quad \text{for} \quad 1 \leq j \leq \nu$$

and

$$W_{jj} = W^*_{jj} \quad \text{for} \quad j > 2\nu.$$

The correspondence $S \to (A, X \bmod \mathcal{W})$ is $1 \to \nu!(N-2\nu)!2^\nu$. From (1.4) we get

$$d\mu_L(S) = \prod_i dz_i \prod_{i \neq j} [dR, A]_{ij}, \qquad (3.4)$$

where $dR = X^{-1}dX$ or

$$d\mu_L(S) = \prod_i dz_i \prod_{i<j} |z_i - z_j|^2 \prod_{i \neq j} dR_{ij}. \qquad (3.5)$$

Therefore

$$P_N^\nu(z_1, \cdots, z_N) = [\nu!(N-2\nu)!2^\nu]^{-1} \prod_{i<j} |z_i - z_j|^2 J, \qquad (3.6)$$

where

$$J = \int \prod_{i \neq j} dR_{ij} \exp[-\mathrm{Tr}\, S^\dagger S]. \qquad (3.7)$$

We define a measure on the set $\mathfrak{X}$ of the X which satisfy (3.3) by

$$d\mu_0(X) = \prod_{i,j} dR_{ij}; \qquad (3.8)$$

$\mathfrak{X}$ is not a group. However the relation: $X_1 \simeq X_2$ iff $X_1^{-1}X_2 \in \mathcal{W}$ is an equivalence relation in $\mathfrak{X}$. We pick one element X_0 in each class. Then any $X \in \mathfrak{X}$ can be written as $X = X_0 W$ in one and only one way. Then

$$dR = X^{-1}\, dX = W^{-1}\, dW + W^{-1}X_0^{-1}\, dX_0\, W \qquad (3.9)$$

implies

$$d\mu_0(X) = d\mu_0(W) \prod_{i \neq j} dR_{ij}, \qquad (3.10)$$

where

$$d\mu_0(W) = \prod_{j=1}^{N} (W^{-1}\, dW)_{jj}$$

is the invariant measure on $\mathcal{W}$. $\mathcal{W}$ is the direct product of the subgroup $\mathcal{U}_0 \subset \mathcal{W}$ of matrices U_0 which satisfy

$$(U_0)_{2j-1,2j-1} = (U_0)^*_{2j,2j} = e^{i\theta_j} \quad \text{for} \quad 1 \leq j \leq \nu$$

and

$$(U_0)_{jj} = \pm 1 \quad \text{for} \quad j > 2\nu$$

and the subgroup $\mathcal{V}$ of real positive matrices $V \in \mathcal{W}$:

$$V_{2j,2j} = V_{2j-1,2j-1} \quad \text{for} \quad 1 \leq j \leq \nu,$$

$$V_{jj} > 0 \quad \text{for} \quad 1 \leq j \leq N.$$

Furthermore,

$$d\mu_0(W) = d\mu_0(V)\, d\mu_0(U_0), \qquad (3.1)$$

where

$$d\mu_0(U_0) = \prod_{j=1}^{\nu} d\theta_j, \qquad (3.12)$$

$$d\mu_0(V) = \prod_{j=1}^{\nu} 2V_{2j,2j}^{-1}\, dV_{2j,2j} \prod_{j=2\nu+1}^{N} V_{jj}^{-1}\, dV_{jj} \qquad (3.13)$$

are the invariant measures on $\mathcal{U}_0$ and $\mathcal{V}$.

Note also that the sum over the ± 1 of the $N - 2\nu$ last diagonal elements of $\mathcal{U}_0$ will result in a factor $2^{N-2\nu}$ in the volume of this group.

We next introduce a representation where only real matrices appear. Let ξ be the unitary matrix

$$\xi = \bigoplus_1^\nu \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \oplus I_{N-2\nu} \qquad (3.14)$$

where $\oplus$ means direct sum. Then $X' = X\xi$ is real. Furthermore

$$d\mu_0(X') = \prod_{i,j} (X'^{-1}\, dX')_{ij} = d\mu_0(X). \qquad (3.15)$$

X' can be written in one and only one way as $X' = OYV'$ where O, Y, V' are real, O is orthogonal (proper or not), Y is triangular (i.e., $Y_{ij} = 0$ for $i > j$) with $Y_{ii} = 1$, and V' is real > 0 and diagonal. Then it follows from

$$X'^{-1}\, dX' = V'^{-1}\, dV' + V'^{-1}Y^{-1}\, dY\, V' + V'^{-1}Y^{-1}(O^{-1}\, dO)YV' \qquad (3.16)$$

and from (3.15) that

$$d\mu_0(X) = d\mu_0(O)\, d\mu_0(Y)\, d\mu_0(V') \qquad (3.17)$$

where

$$d\mu_0(O) = \prod_{i<j} (O^{-1}\, dO)_{ij}, \qquad (3.18)$$

$$d\mu_0(Y) = \prod_{i<j} (Y^{-1}\, dY)_{ij} = \prod_{i<j} dY_{ij}, \qquad (3.19)$$

$$d\mu_0(V') = \prod_i V_{ii}'^{-1}\, dV'_{ii}. \qquad (3.20)$$

V' can be decomposed in one and only one way as $V' = VT$ where $V \in \mathcal{V}$ and

$$T = \bigoplus_1^\nu \begin{pmatrix} t_j & 0 \\ 0 & 1/t_j \end{pmatrix} \oplus I_{N-2\nu}, \qquad (3.21)$$

with

$$d\mu_0(V') = d\mu_0(V)\, d\mu_0(T), \qquad (3.22)$$

where $d\mu_0(V)$ is defined in (3.13) and

$$d\mu_0(T) = \prod_{i=1}^{\nu} dt_i/t_i.$$

We now come back to (3.7). Now Tr $S^\dagger S =$ Tr $A^\dagger HAH^{-1}$, where H should be $X^\dagger X = \xi X'^\dagger X'\xi^\dagger$. However, due to $X' = OYVT$ and the fact that V commutes with both ξ and A, one can take $H = \xi H'\xi^\dagger$, where

$$H' = TY^\dagger YT \tag{3.23}$$

depends only on Y and T. We now introduce in (3.7) an extra integration over $\mathfrak{U}_0$ and make use of (3.10), (3.11), (3.17), and (3.22). We obtain

$$J = \frac{\Omega_0}{(2\pi)^\nu 2^{N-2\nu}} \times \int \exp\left[-\mathrm{Tr}\, S^\dagger S\right] d\mu_0(Y)\, d\mu_0(T), \tag{3.24}$$

where Ω_0 is the volume of the full orthogonal group and $(2\pi)^\nu 2^{N-2\nu}$ is the volume of $\mathfrak{U}_0$. Now it follows from (3.23) that

$$d\mu_0(Y) = \prod_{i<j} dY_{ij} = \prod_{i<j} d(Y^\dagger Y)_{ij} = \prod_{i<j} dH'_{ij}, \tag{3.25}$$

the Jacobian of the last transformation being $(\det T)^{2\nu-1} = 1$. Finally,

$$J = \frac{\Omega_0}{(2\pi)^\nu 2^{N-2\nu}} \times \int \exp\left[-\mathrm{Tr}\, S^\dagger S\right] \prod_{i<j} dH'_{ij} \prod_{j=1}^{\nu} dt_j/t_j. \tag{3.26}$$

It follows from (3.23) that the $2j-1 \times 2j-1$ (or $2j \times 2j$) upper left blocks of H' have determinant t_j^2 (or 1) for $j = 1, \cdots, \nu$ and that the $j \times j$ upper left blocks have determinant 1 for $j \geq 2\nu$. Independent variables are therefore the H'_{ij} with $i < j$, or $i = j = 2k - 1$ for $k = 1, \cdots, \nu$. The other diagonal elements of H' are determined in term of them by the previous condition.

We now come back to the original complex representation and express the differential element in (3.26) in terms of H alone. One can replace $\prod dH'_{ij}$ by the corresponding $\prod dH_{ij}$ except perhaps for the first 2×2 blocks along the diagonal.

If $\binom{a\ b}{b\ c}$ is the jth such block in H', the corresponding block in H is $\binom{u\ \zeta}{\zeta^*\ u}$, where $u = \frac{1}{2}(a + c)$, $\zeta = \frac{1}{2}(a - c) + ib$. From (3.23) it follows that $a = t_j^2 a'$ and $c = t_j^{-2}c'$, where a', c' depend only on Y, as well as b. In particular, for fixed Y:

$$2\frac{dt_j}{t_j} = \frac{da}{a} = -\frac{dc}{c} = \frac{d(a-c)}{a+c}.$$

The contribution of this 2×2 block to the differential element in (3.26) is then easily seen to be $db\, dt_j/t_j = d\zeta\, d\zeta^*/4u$. Therefore

$$J = \frac{\Omega_0}{2^N(2\pi)^\nu} \int \exp\left[-\mathrm{Tr}\, S^\dagger S\right] d\mu(H), \tag{3.27}$$

where

$$d\mu(H) = \prod_{i<j} dH_{ij} \prod_{j=1}^{\nu} (H_{2j,2j})^{-1}\, dH_{2j,2j-1}. \tag{3.28}$$

We can now integrate over the last $N - 2\nu$ rows and columns of H by the same induction procedure as in Sec. 1. We use (1.28) for $n = N$ (z_n real). The integration over the last row and column is then straightforward and gives a factor

$$\exp\left[-|z_N|^2\right]\pi^{(N-1)/2}\Big[\prod_{i<N}|z_N - z_i|\Big]^{-1}.$$

After $N - 2\nu$ similar steps, we obtain

$$J = \frac{\Omega_0 \pi^{(N-2\nu)(N+2\nu-1)/4}}{2^N(2\pi)^\nu} \times \Big[\prod_{2\nu<i<j\leq N}|z_i - z_j| \prod_{l\leq 2\nu<j}|z_l - z_j|\Big]^{-1} \times \exp\Big[-\sum_{j>2\nu}|z_j|^2\Big]F_\nu(z_1, \cdots, z_{2\nu}), \tag{3.29}$$

where $F_\nu(z_1, \cdots, z_{2\nu})$ is the value of the integral $\int \exp[-\mathrm{Tr}\, S^\dagger S]\, d\mu(H)$ when S is a $2\nu \times 2\nu$ matrix *without* real eigenvalues for $\nu > 0$, and $F_0 = 1$. Substituting (3.29) into (3.6), we obtain

$$P_N^\nu(z_1, \cdots, z_N) = \frac{\Omega_0\pi^{(N-2\nu)(N+2\nu-1)/4}}{\nu!(N-2\nu)!(4\pi)^\nu 2^N}\prod_{i<j}|z_i - z_j| \times \exp\Big[-\sum_{i>2\nu}|z_i|^2\Big]\prod_{i<j\leq 2\nu}|z_i - z_j|\, F_\nu(z_1, \cdots, z_{2\nu}). \tag{3.30}$$

In particular, for $\nu = 0$, all the eigenvalues are real: $z_i = r_i$ $(i = 1, \cdots, N)$,

$$P_N^0(r_1, \cdots, r_N) = \frac{\Omega_0\pi^{N(N-1)/4}}{2^N N!}\prod_{i<j}|r_i - r_j| \times \exp\Big[-\sum_{i=1}^{N} r_i^2\Big]. \tag{3.31}$$

For $\nu \neq 0$, F_ν cannot be calculated by the previous method. One can obtain a recursion formula analogous to (1.28), though somewhat less simple. However, the integration over the last two rows and columns of H does not lead to elementary functions as previously and the induction procedure breaks down completely.

Therefore we are able to determine P_N explicitly only in the particular case where all eigenvalues are real. In the general case, we have extracted the dependence of P_N on the real eigenvalues, and the complex eigenvalues still appear in a factor F, for which we have only an intractable integral representation.

CONCLUSION

The results of our investigation are essentially contained in Eqs. (1.35), (1.40), and (1.42) for the complex case; (2.27) for the quaternion case; (3.30) for the real case. They are remarkably simple in the complex case, where, in particular, the eigenvalue density tends to a constant $\rho_N(z) \rightarrow \rho = 1/2\pi$ as $N \rightarrow \infty$. The quaternion case involves essentially technical problems. In the real case, however, one meets major difficulties which seem to come from the fact that the real field is not algebraically closed. We have considered here only a restricted class of ensembles. One could generalize by keeping the same algebraic set of matrices and defining other measures $d\mu'$ instead of $d\mu$ defined in (0.1). For instance, if $d\mu'(S) = \mathcal{P}(S)d\mu(S)$, where $\mathcal{P}$ is a polynomial in Tr (S^r) $(r = 1, \cdots, N)$ (if $r' > N$, $\operatorname{Tr} S^{r'}$ is a polynomial in $\operatorname{Tr} S^r$, $r \leq N$), then nothing is changed in the calculation of P_N, because $\mathcal{P}(S)$ depends only on A. However the induction procedure by which we performed the integration on X (or H) seems to be a very specific property of $\exp[-\operatorname{Tr} S^\dagger S]$.

One could also consider other algebraic sets of matrices, for instance Dyson's V ensembles (5) with measures analogous to (0.1). Nothing new appears in the complex case where V is identical with Z. In the real case, V is the set of complex symmetric matrices; the calculation of P_N seems to be extremely complicated, and the simplest case of 2×2 matrices is not encouraging. In the quaternion case, we have not reached any conclusion.

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