

Chapter 1

Experiments with light

1.1 Electromagnetic waves

According to Maxwell light is an electromagnetic wave of electric $\mathbf{E}(\mathbf{x}, t)$ and magnetic $\mathbf{B}(\mathbf{x}, t)$ fields freely oscillating in vacuum. The solutions of Maxwell's equations in empty space are superpositions of monochromatic modes of frequency ω . A mode, or plane wave, propagating along the z axis, is given by

$$\mathbf{E}(\mathbf{x}, t) = \text{Re } \mathbf{E}_0 e^{i(kz - \omega t)}, \quad \mathbf{B}(\mathbf{x}, t) = \frac{1}{c} \hat{\mathbf{z}} \times \mathbf{E}(\mathbf{x}, t), \quad \omega = ck \quad (1.1)$$

The amplitude vector \mathbf{E}_0 (thus \mathbf{E} and \mathbf{B} also) always belongs to the $(x, y) \perp z$ plane,

$$\mathbf{E}_0 = E_0 \begin{bmatrix} \cos \theta e^{i\delta_x} \\ \sin \theta e^{i\delta_y} \\ 0 \end{bmatrix} \quad (1.2)$$

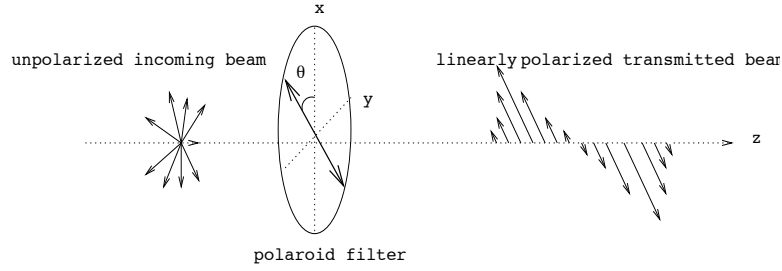
The energy per unit time per unit surface that would be imparted to a material object by the wave, is given by the norm of the Poynting vector

$$\mathbf{S} = \epsilon_0 c^2 \mathbf{E} \times \mathbf{B} \quad (1.3)$$

A convenient measure of the *intensity* I of the wave is given by the norm of \mathbf{S} averaged over a period $T = \frac{2\pi}{\omega}$,

$$I = \frac{1}{2} \epsilon_0 c |\mathbf{E}_0|^2 = \frac{1}{2} \epsilon_0 c E_0^2 \quad (1.4)$$

From (1.1), (1.2) it follows in general that the tip of the electric (and hence also magnetic) field vector describes, as a function of time, an ellipse

Figure 1.1: Preparation of beam polarized along θ

in the (x, y) plane. There are two degenerate cases of special importance. *Linear polarization* corresponds to $\delta_x - \delta_y = m\pi$ (m integer) and the tip of the field oscillates in the (x, y) plane on a line making the angle θ with x (m even/odd). For $\theta = \frac{\pi}{4}$ (so that $\cos \theta = \sin \theta = \frac{1}{\sqrt{2}}$) and $\delta_x - \delta_y = m\frac{\pi}{2}$ (m odd integer) the polarization is left/right *circular* which means that the tip of the field rotates along a circle of radius E_0 . A light beam can be easily prepared in a state of linear polarization with the help of a filter which transmits only the component of the electric field along θ . All our subsequent discussion does not rely on a detailed explanation of the phenomenon and we do not need to know more about it¹. Such a device is called a *polarizer* with axis θ (figure 1.1).

Assume that a source of light has been prepared in a state of linear polarization along θ as in figure 1.1.

$$\mathbf{E}_{\text{in}}(\mathbf{x}, t) = E_0 \begin{bmatrix} \cos \theta \\ \sin \theta \\ 0 \end{bmatrix} \text{Re} e^{i(kz - \omega t)} \quad (1.5)$$

The intensity of the prepared beam (1.5) is proportional to E_0^2 . Suppose now that this ray is transmitted through a second polarizer at an angle α . This second polarizer is called the *analyzer*. The light is then collected by a detector² and its intensity measured (see figure 1.2). The electric field of the final beam is obtained by projecting the incoming electric field on the

¹In fact so-called absorptive polarizers are made of sheets of anisotropic crystals allowing electron motion preferentially in the θ_{\perp} direction. The θ_{\perp} component of the electric field sets electrons into a state of oscillation which produces the emission of an emitted antiphase electromagnetic wave polarized along θ_{\perp} . The later cancels the progressive θ_{\perp} component of the wave so that the net effect is to leave out a θ transmitted component and a θ_{\perp} reflected component.

²This can be a photoelectric cell which transforms the electromagnetic energy into a current.

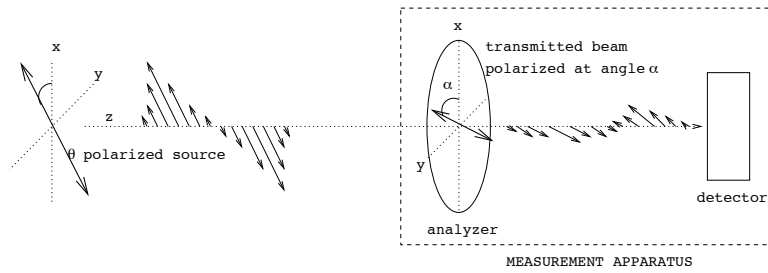


Figure 1.2: analyzer-detector measurement apparatus

analyzer axis \mathbf{e}_α

$$\mathbf{E}_{\text{out}} = (\mathbf{E}_{\text{in}} \cdot \mathbf{e}_\alpha) \mathbf{e}_\alpha = E_0 \cos(\theta - \alpha) \begin{bmatrix} \cos \alpha \\ \sin \alpha \\ 0 \end{bmatrix} \text{Re} e^{i(kz - \omega t)} \quad (1.6)$$

and the intensity received in D is proportional to $E_0^2 \cos^2(\theta - \alpha)$. To summarize, when a beam polarized along θ is transmitted through an analyzer at an angle α , the outgoing beam is polarized along α and the fraction of intensity collected by the detector (average power per unit surface) is³

$$\frac{I_{\text{out}}}{I_{\text{in}}} = \cos^2(\theta - \alpha) \quad (1.7)$$

In particular if $\alpha - \theta = 0, \pi$ all the light passes through the analyzer, while if $\alpha - \theta = \pm \frac{\pi}{2}$ none of it is transmitted. The analyzer-detector system can be used as a measurement apparatus to determine the polarization of a wave (assuming we know a priori that it is linear) by adjusting the angle α such that the collected intensity varies from 0 to its maximal value. Let us now describe two simple experiments with electromagnetic waves.

Polarizing beamsplitter experiment. There exist prisms⁴ that have the property of splitting a beam in two linearly polarized ones, one is polarized perpendicular to the incidence plane while the other is polarized parallel to that plane. In figure 1.3 the incidence plane is (x, z) so one ray has y polarization while the other one has x polarization. Two detectors D_x and D_y measure the outgoing intensities of each beam. Note that the polarization degree of freedom is coupled to the orbital (path of ray) degree of freedom.

³Malus law.

⁴These are made of quartz or calcite crystals whose refraction index are different for polarization perpendicular to, versus into, the incidence plane. Such crystals are called birefringent, one ray is called ordinary because the direction of refraction obeys the usual Snell law, while the other ray is called extraordinary.

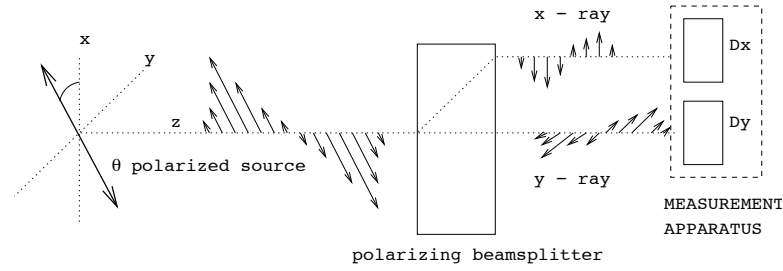


Figure 1.3: polarizing beamsplitter experiment

Before the polarizing beam splitter the electric field is given by (1.5) and has intensity proportional to E_0^2 . After the beam splitter the x -polarized ray has an electric field

$$\mathbf{E}_x = E_0 \begin{bmatrix} \cos \theta \\ 0 \\ 0 \end{bmatrix} \text{Re } e^{i(kz - \omega t)} \quad (1.8)$$

and the intensity detected at D_x is proportional to $E_0^2 \cos^2 \theta$, while the y -polarized ray has a field

$$\mathbf{E}_y = E_0 \begin{bmatrix} 0 \\ \sin \theta \\ 0 \end{bmatrix} \text{Re } e^{i(kz - \omega t)} \quad (1.9)$$

and its intensity measured by D_y is proportional to $E_0^2 \sin^2 \theta$. Both detectors collect a fraction of the intensity,

$$\frac{I_{\text{out},x}}{I_{\text{in}}} = \cos^2 \theta, \quad \frac{I_{\text{out},y}}{I_{\text{in}}} = \sin^2 \theta \quad (1.10)$$

In this experiment absorption and reflection by the prism are negligible so that the sum of these two fractions equals 1.

Decomposition-recombination experiment. Once we have decomposed light with a polarizing beamsplitter, we can recombine it with a symmetric prism. We analyze the recombined beam with an analyzer-detector apparatus (see figure 1.4). Let us carefully review the situation. Before the first beamsplitter we have *one ray* with electric field given by (1.5). The first beamsplitter splits the ray in *two parts* with electric fields given by (1.8) and (1.9). After the second beamsplitter the two rays *interfere* and the electric field of the recombined beam is the sum of (1.8) and (1.9), which equals (1.5). *The fraction of intensity collected by the analyzer-detector system is*

$$\cos^2(\theta - \alpha) \quad (1.11)$$

a fact consistent with the first experiment.

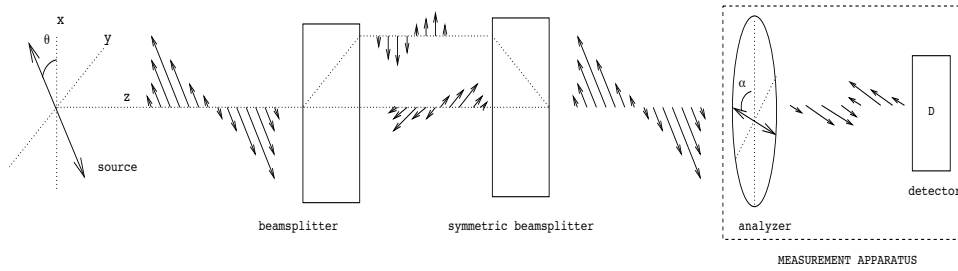


Figure 1.4: decomposition-recombination experiment

1.2 Photons

Einstein's explanation of the photoelectric effect, assumes that the interaction of light with matter occurs through discrete quanta (quantities) of energy and momentum that are absorbed and emitted. These quanta are called photons, and each photon carries an energy $\hbar\omega$ and momentum $\hbar k$ (where $\omega = ck$ still holds). If we think of the beam as a collection of independent photons, its intensity is $\hbar\omega c \frac{N}{V}$ where $\frac{N}{V}$ is the number of photons per unit volume⁵. Identifying this quantity with (1.4) we find a relation between the electric field and the number of photons associated to the electromagnetic wave.

If we diminish sufficiently the intensity of the source we arrive at a situation where in principle photons are emitted one by one. We will repeat the experiments with such a *single photon source*, that prepares them in a state of polarization θ .

Let us first discuss how the analyzer-detector measurement apparatus works. We repeat the experiment of figure 1.2 and collect photons at the detector D . When a photon hits the detector the later clicks (an electric pulse is triggered) - we record this event as a 1, otherwise we record 0. This experiment produces a sequence

$$1001111000101010011101\dots \quad (1.12)$$

that looks random and where the empirical fraction of 1's is $\cos^2(\theta - \alpha)$. From this experiment we infer

$$\text{probability of detecting a photon} = \cos^2(\theta - \alpha) \quad (1.13)$$

In particular if $\alpha - \theta = 0, \pi$ all photons are detected while if $\alpha - \theta = \pm \frac{\pi}{2}$ no photon is detected.

⁵ $c \frac{N}{V}$ is the number of photons par unit time per unit surface that hit a detector.

This experiment shows that photons carry a polarization. However we already see that they behave in a peculiar way, indeed for general angles it is not possible to predict the precise outcome for each individual photon. We will see that this is a fundamental feature of the measurement process for quantum systems and that it is not at all obvious to reconcile this fact with our classical intuitions. One could attempt a classical interpretation⁶ - which is wrong - by saying that the photon is particle-like object that undergoes complicated but otherwise deterministic collision processes within the analyzer, which result in a probability $\cos^2(\theta - \alpha)$ of being transmitted. In view of (1.7) and (1.13) this idea even seems compatible with the picture of a wave being a collection of independent photons.

Let us now repeat the two previous experiments with photons that are sent one by one.

Polarizing beamsplitter experiment. Each single photon (polarized at an angle θ) goes through the prism. We observe that either D_x clicks (the upper detector register a 1 and the lower a 0) or D_y clicks (the upper detector registers a 0 and the upper a 1); but they never click simultaneously. We record two random complementary sequences with respective fractions of 1 equal to $\cos^2 \theta$. Empirically,

$$\text{prob detect photon at } D_y = \sin^2 \theta, \quad \text{prob detect photon at } D_x = \cos^2 \theta \quad (1.14)$$

The sum is equal to one which means that the photon has certainly passed through the beamsplitter.

The fact that the detectors never click at the same time, contrarily to what happens with waves, suggests the same (wrong) classical interpretation, alluded to above. A photon is a *sort of particle*, which due to complicated but otherwise deterministic collisions with the crystal, is deflected towards the lower path with probability $\sin^2 \theta$ or through the upper path with probability $\cos^2 \theta$. This seems again consistent with the observed intensities of the experiment with waves if we assume the wave is a collection of photons.

Decomposition-recombination experiment. let us consider again the setting of figure 1.4. When photons are sent one by one we again record a sequence of random clicks, and we infer from this sequence

$$\text{prob detect photon at } D = \cos^2(\theta - \alpha) \quad (1.15)$$

This should come as a great surprise to the reader. Indeed this result is not consistent with the particle-like picture of a photon, but rather with a wave-like picture, as we now show.

⁶in the spirit of statistical mechanics, say

If a photon takes the lower path in figure 1.4 its polarization is horizontal before the second beamsplitter and comes out of it in a horizontal state. Therefore the probability of transmission of such a lower-path photon through the analyzer is $\cos^2(\frac{\pi}{2} - \alpha) = \sin^2 \alpha$. Therefore

$$\text{prob(D clicks | lower path)} = \sin^2 \alpha \quad (1.16)$$

If the photon takes the upper path its polarization is vertical just before the second beamsplitter and comes out in a state of vertical polarization. Therefore the probability of transmission of such an upper-path photon is $\cos^2(0 - \alpha)$ and

$$\text{prob(D clicks | upper path)} = \cos^2 \alpha \quad (1.17)$$

Now, we have

$$\begin{aligned} \text{prob(D clicks)} = & \text{prob(D clicks | lower path)}\text{prob(lower path)} \\ & + \text{prob(D clicks | upper path)}\text{prob(upper path)} \end{aligned} \quad (1.18)$$

Thus because of (1.14), (1.16), (1.17)

$$\text{prob detect photon at D} = \sin^2 \theta \sin^2 \alpha + \cos^2 \theta \cos^2 \alpha \quad (1.19)$$

This contradicts the experimental result (1.15) and is therefore wrong ! The term that is missing is precisely

$$2 \cos \theta \cos \alpha \sin \theta \sin \alpha \quad (1.20)$$

which, in wave theory, appears because of the *interference* between the x and y components of the electric field.

We thus face the following situation. The decomposition experiment 1.3 suggests that photons behaves in a *particle-like* manner (like a billiard ball) choosing the upper or lower path according to some underlying complicated collision process. The recombination experiment (1.4) suggests that photons behave in a *wave-like* fashion. As for most dilemmas, the resolution offered by quantum theory teaches us that both pictures are a partial reflection of a more subtle reality.

1.3 The quantum setting: first encounter

In fact all known forms of matter⁷ display this particle/wave duality. As we will now see quantum mechanics offers us a picture which accomodates both

⁷For example photons, electrons, nuclei and their constituents ...

behaviors and superseedes the classical pictures of wave and particle⁸.

We will illustrate how the rules of quantum theory consistently explain the three experiments. The situation will be modelled in the simplest possible way which retains the basic essence of quantum mechanics.

The state of a photon is described by two degrees of freedom, an *orbital* degree of freedom and a *polarization* degree of freedom. Let us first concentrate on polarization. The state of polarization is described by a unit vector \mathbf{e} perpendicular to the direction of motion. Following Dirac we call these state vectors *kets* and denote them as $|\mathbf{e}\rangle$. Since the polarization vector lies in the x, y plane it can be described in a orthonormal basis $|\uparrow\rangle, |\leftrightarrow\rangle$, corresponding to the two linear states of polarization along x and y

$$|\mathbf{e}\rangle = \lambda|\uparrow\rangle + \mu|\leftrightarrow\rangle, \quad |\lambda|^2 + |\mu|^2 = 1 \quad (1.21)$$

A state of linear polarization along θ corresponds to $\lambda = \cos \theta$ and $\mu = \sin \theta$, so that (1.21) becomes

$$|\theta\rangle = \cos \theta|\uparrow\rangle + \sin \theta|\leftrightarrow\rangle \quad (1.22)$$

On the other hand for circular polarization the x and y components of the polarization vector have a $\frac{\pi}{2}$ - phase difference, thus

$$|L/R\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \pm i|\leftrightarrow\rangle) \quad (1.23)$$

Given a state vector $|\Phi\rangle$ its *adjoint* (also called hermitian conjugate) is obtained by taking the complex conjugate and transposing $\overline{|\Phi\rangle}^T$. This is denoted as a *bra*

$$\langle\Phi| = \overline{|\Phi\rangle}^T \quad (1.24)$$

The usual inner product (defined over a complex vector space) is called the *bracket*

$$\langle\Phi'|\Phi\rangle = (\overline{|\Phi'\rangle}^T) \cdot (|\Phi\rangle) \quad (1.25)$$

As an example consider the inner product between two polarization state vectors. First the conjugate of a linearly polarized state is

$$\langle\alpha| = \langle\uparrow| \cos \alpha + \langle\leftrightarrow| \sin \alpha \quad (1.26)$$

⁸According to modern physics, matter is described by relativistic quantum fields. There are underlying quantum fields (e.g. the quantum electromagnetic field, the quantum electronic field, the quark field ect...) which may manifest themselves in a wave-like or particle-like fashion depending on the situation. We will not introduce field theoretical notions in this course.

The inner product with $|\theta\rangle$ then is

$$\begin{aligned}\langle\alpha|\theta\rangle &= (\langle\uparrow|\cos\alpha + \langle\leftrightarrow|\sin\alpha) \cdot (\cos\theta|\uparrow\rangle + \sin\theta|\leftrightarrow\rangle) \\ &= \cos\alpha\cos\theta + \sin\alpha\sin\theta \\ &= \cos(\theta - \alpha)\end{aligned}\tag{1.27}$$

To obtain the second equality one expands the braces into four terms, uses linearity of the bracket and the orthonormality condition,

$$\langle p|p'\rangle = \delta_{pp'}\tag{1.28}$$

It is instructive to check that the circularly polarized states ($|L\rangle, |R\rangle$) form another orthonormal basis.

Let us now introduce the orbital degree of freedom in the picture. For a freely moving photon, i.e a photon that does not interact with a material object, the orbital state is entirely described once we know its *momentum* \mathbf{k} , which has a *direction* and a *norm* $k = \frac{\omega}{c}$. The state vector is now denoted as $|\mathbf{k}, \mathbf{e}\rangle$. This state freely evolves with time and for a photon of frequency ω the time evolution simply amounts to a multiplicative phase factor, which does not change the momentum and the polarization. The photon state at time t is

$$|\Psi_{\mathbf{k}, \mathbf{e}}(t)\rangle = e^{i\omega t}|\mathbf{k}, \mathbf{e}\rangle\tag{1.29}$$

An explanation is order here about the kets indexed by two degrees of freedom. We will see in the next chapter that the mathematical rule to combine degrees of freedom is the tensor product; this means that $|\mathbf{k}, \mathbf{e}\rangle = |\mathbf{k}\rangle \otimes |\mathbf{e}\rangle$ and that the inner product is

$$\langle\mathbf{k}', \mathbf{e}'|\mathbf{k}, \mathbf{e}\rangle = \langle\mathbf{k}'|\mathbf{k}\rangle \cdot \langle\mathbf{e}'|\mathbf{e}\rangle\tag{1.30}$$

Finally the momentum vectors themselves form an orthonormal basis $\langle\mathbf{k}'|\mathbf{k}\rangle = \delta_{\mathbf{k}', \mathbf{k}}$.

As we will see in the next chapter, in general, the time-evolution of isolated systems is given by a unitary transformation. In (1.29) the unitary transformation is simply the multiplication by the phase factor. When the photon interacts with matter (for example with the analyzer, the beamsplitter) one has in principle to describe the unitary evolution of the total system (photon + analyzer or photon + beamsplitter), which is then more complicated. Here we do not have to discuss such issues as we consider only the in-going and out-going states which are those of freely moving photons.

When we make a measurement on a system, the system that is observed cannot be considered as isolated and the state is modified in a non-unitary

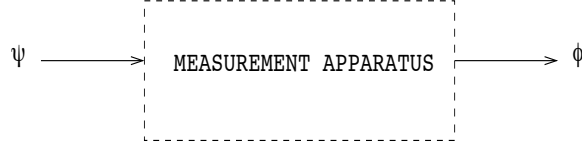


Figure 1.5: measurement with initial state $|\Psi\rangle$ and outcome $|\Phi\rangle$.

way. Explaining the measurement process is a subject that has been (and sometimes is still) much debated since the early days of quantum mechanics. To this day, the only fully consistent operational rule, to determine the outcome of a measurement is given by the so-called *measurement postulate* in the form advocated by the Copenhagen School and notably Niels Bohr⁹ (figure 1.5). Here we give it in a rough form, and will be more precise in the next chapter.

If a system is initially prepared in the state $|\Psi\rangle$ and the outcome of the measurement is a state $|\Phi\rangle$, the probability of this event is

$$\text{Prob}(|\Psi\rangle \rightarrow |\Phi\rangle) = |\langle\Phi | \Psi\rangle|^2 \quad (1.31)$$

One cannot predict the outcome but only its frequency of occurrence during repeated identical experiments with identical initial states.

The transition between the initial and final state is also called "reduction" or "collapse" of the state. We will see in the next chapter that given a measurement apparatus, or a quantity that is being measured, there are a number of possible outcomes and that the transition probabilities sum to one.

The re-interpretation of the three experiments in the next section should make this rather abstract postulate a bit more natural.

1.4 Quantum interpretation of experiments.

We assume that the source prepares single photons in the linearly polarized, freely moving state

$$|\Psi_{\mathbf{k},\theta}(t)\rangle = e^{i(kz - \omega t)} |\mathbf{k}, \theta\rangle \quad (1.32)$$

If the measurement apparatus is the analyzer-detector system of figure 1.2, the measurement postulate tells us that the probability to find the photon

⁹Einstein never agreed that one could not predict the outcome and thought that the probabilities are an effective description. In his words "God does not play with dice".

in state $|\mathbf{k}, \alpha\rangle$ is

$$|\langle \mathbf{k}, \alpha | \Psi_{\mathbf{k}, \theta}(t) \rangle|^2 = |\langle \alpha | \theta \rangle|^2 = \cos^2(\theta - \alpha) \quad (1.33)$$

This is consistent with the experimentally measured frequency of clicks in D .

Polarizing beamsplitter experiment. Before the beam splitter the photon state is (1.32), which can be written as

$$e^{i(kz - \omega t)} (\cos \theta |\mathbf{k}, \uparrow\rangle + \sin \theta |\mathbf{k}, \leftrightarrow\rangle) \quad (1.34)$$

After the beamsplitter it becomes

$$e^{i(kz - \omega t)} (\cos \theta |\mathbf{k}_u, \uparrow\rangle + \sin \theta |\mathbf{k}_l, \leftrightarrow\rangle) \quad (1.35)$$

where \mathbf{k}_u and \mathbf{k}_l label the upper and lower paths¹⁰. Notice that contrary to (1.34), in (1.35) we cannot separate the orbital and polarization degrees of freedom into a tensor product: it can be shown that for (1.35) this is an intrinsic property that does not depend on the basis. We say that the orbital and tensor degrees of freedom have been *entangled* by the beamsplitter. Entangled states depart fundamentally from the classical picture and retain quantum correlations that are missing in the classical interpretation. As we will see in this course they play a very important role in quantum information and computation because they may offer resources that are non-classical.

Now we consider the two detectors as our measurement apparatus. The measurement postulate tells us that the probability to observe the photon in state $|\mathbf{k}_u, \uparrow\rangle$ is

$$|\langle \mathbf{k}_u, \uparrow | e^{i(kz - \omega t)} (\cos \theta |\mathbf{k}_u, \uparrow\rangle + \sin \theta |\mathbf{k}_l, \leftrightarrow\rangle) \rangle|^2 = \cos^2 \theta \quad (1.36)$$

Similarly the probability to observe it in the state $|\mathbf{k}_l, \leftrightarrow\rangle$ is

$$|\langle \mathbf{k}_l, \leftrightarrow | e^{i(kz - \omega t)} (\cos \theta |\mathbf{k}_u, \uparrow\rangle + \sin \theta |\mathbf{k}_l, \leftrightarrow\rangle) \rangle|^2 = \sin^2 \theta \quad (1.37)$$

This is consistent with the experimental fractions of clicks at D_x and D_y with probability $\cos^2 \theta$ and $\sin^2 \theta$.

Recombination experiment. The second polarizing beamsplitter transforms the entangled state (1.35) back to (1.32). The later state enters the measurement apparatus constituted by the analyzer-detector system. Therefore the probability of observing $|\mathbf{k}, \alpha\rangle$ is simply given by (1.33) and we correctly recover. This is the experimental frequency of clicks at D ! The quantum interpretation does not lose track of the interference term (1.20).

¹⁰Here we may imagine that the paths are not quite in the same direction so that these two labels are different. In principle one should make a more complete description of the orbital part of the state that takes into account the finite width of the beams.

1.5 Notion of quantum bit

There exist many quantum systems in nature that can be described by state vectors which belong to the vector space \mathbf{C}^2 , the two dimensional complex vector space. If we call $|0\rangle$ and $|1\rangle$ two orthonormal basis states a general state vector takes the form

$$|\psi\rangle = \lambda|0\rangle + \mu|1\rangle, \quad |\lambda|^2 + |\mu|^2 = 1 \quad (1.38)$$

It will often be convenient to identify

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (1.39)$$

and

$$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (1.40)$$

and in quantum information theory it is customary to call this canonical basis the *computational basis*. Of course one can represent the quantum bit $|\psi\rangle$ in any other basis, and one that we will often use one that is obtained by a standard 45 degree real rotation

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \quad (1.41)$$

This basis will be called the Hadamard basis. Since the vector space is complex we can make more general unitarity transformations. For example

$$|L\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle), \quad |R\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle) \quad (1.42)$$

We have already seen a physical realization of a quantum bit, namely the photon polarization. If we identify the computational basis with horizontal/vertical polarized photon states, then the Hadamard basis corresponds to polarized states at 45 degree angle, and the last basis obtained by a unitary transformation is physically realized by circularly left/right polarized photons. A physically meaningful parametrization of general polarization state is

$$|\psi\rangle = e^{i\delta_x} \cos \theta |\uparrow\rangle + e^{i\delta_y} \sin \theta |\leftrightarrow\rangle \quad (1.43)$$

This is a vector representation of the two dimensional rotation group around the axis of propagation (z axis say). This means that we rotate our reference frame (around z) by angle β , then the state vector is obtained from the above

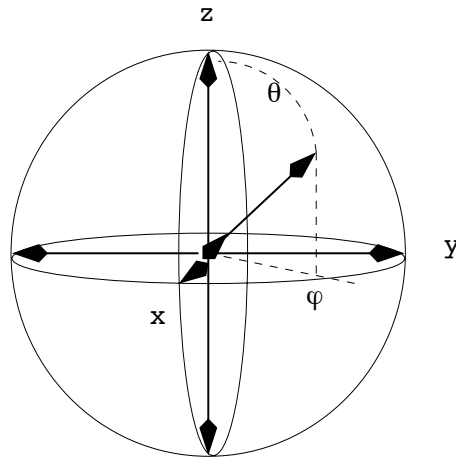


Figure 1.6: Bloch sphere. Computational (z), Hadamard (x), circular (y) basis states

expression by the replacement $\theta \rightarrow \theta - \beta$. In particular if the reference frame is rotated by 2π we recover the same state vector.

Another very common but physically different quantum bit is the *spin* $\frac{1}{2}$. The most famous elementary particle (of obvious importance in our everyday life since it transports electricity, interacts with sunlight ...) that has spin $\frac{1}{2}$ is the electron¹¹. There exist also many composite systems, such as nuclei or atoms that carry a total spin of $\frac{1}{2}$. A very rough intuitive way of thinking about spin is to view the particle (the electron say) as having intrinsic spinning motion around some axis. If the particle spins around the z axis its spin is $|\uparrow\rangle$ or $|\downarrow\rangle$ according to its direction of rotation. These two states form a basis and the most general spin state is

$$|\psi\rangle = \lambda|\uparrow\rangle + \mu|\downarrow\rangle, \quad |\lambda|^2 + |\mu|^2 = 1 \quad (1.44)$$

Spin $\frac{1}{2}$ states are two dimensional complex representations of the three dimensional rotation group. For this reason a physically meaningful parametrization is

$$|\psi\rangle = e^{i\frac{\phi}{2}} \cos\frac{\theta}{2} |\uparrow\rangle + e^{-i\frac{\phi}{2}} \sin\frac{\theta}{2} |\downarrow\rangle \quad (1.45)$$

These states can be represented by the tip of a vector on the *Bloch sphere* (figure 1.6) with the usual spherical coordinates (θ, ϕ) . We have the following correspondence (up to phase factors):

$$\theta = 0, \pi \quad |\uparrow\rangle, |\downarrow\rangle, \quad \text{particle spin along } z \quad (1.46)$$

¹¹Constituents of nuclei, protons and neutrons also have spin $\frac{1}{2}$. In particular the interaction of the proton spin with magnetic fields is at the basis of Nuclear Magnetic Resonance, used for example in medical imagery.

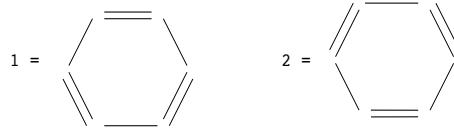


Figure 1.7: possible arrangements of chemical bonds

$$\theta = \frac{\pi}{2}, \phi = 0, \pi \quad |\uparrow\rangle \pm |\downarrow\rangle, \quad \text{particle spin along x} \quad (1.47)$$

$$\theta = \frac{\pi}{2}, \phi = \pm \frac{\pi}{2} \quad |\uparrow\rangle \pm i|\downarrow\rangle, \quad \text{particle spin along y} \quad (1.48)$$

The polarization and spin $\frac{1}{2}$ quantum bits are different representations of the rotation group in quantum mechanics (ultimately coming from the representation of the Lorentz group of relativity). But there exist also other realizations of the quantum bit that have nothing to do with the representations of the rotation group in quantum physics. An example is given by the benzene molecule C_6H_6 that can be in the two states that differ in the arrangement of single and double electronic bonds (figure 1.7). But the molecule can also be found in a *resonating state* such as

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle) \quad (1.49)$$

What is the difference between a bit and a quantum bit? A classical bit is an abstraction of a physical quantity that can be reasonably well described by a two valued quantity. Examples are the charge in a capacitor, a voltage difference, or the magnetization of a Weiss domain. Classical information theory is sufficiently universal so that it does not have to account for the detailed physical properties of the classical bits. The only underlying assumption is that these exist in two definite values 0 or 1 (let us pretend that noise is absent). Suppose a classical bit is given to you and that you have no information whatsoever about its value. To gain information about its value you can observe it (measure the charge, the voltage difference) and its value is then discovered. By *discovered* we mean that it already had the observed value before the measurement, and that the measurement has not destroyed it. In this sense the information that we gain, or the uncertainty that is removed, is by definition 1 bit.

A quantum bit is also an abstraction of physical quantities as the above examples have shown. It is well described by a two dimensional complex vector. In the same spirit than in the classical case, quantum information theory is sufficiently universal so that many of its aspects are independent of the concrete physical realization. However the point is that it takes into

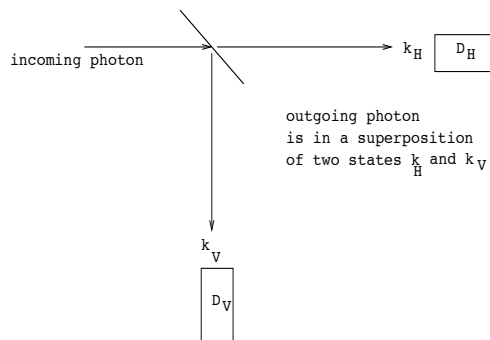


Figure 1.8: semi-transparent mirror

account the general underlying laws of quantum mechanics. This means in particular that extracting information from quantum bits is quite different than in the classical case. Suppose that a quantum bit is given to you in some state $|\psi\rangle$ on which you do not have any information whatsoever. In order to determine $|\psi\rangle$, we have to observe it (agree?). To perform a measurement we have to select an apparatus, in other words an orthonormal basis $|b_1\rangle, |b_2\rangle$. The measurement process then reduces the quantum bit to $|b_1\rangle$ or to $|b_2\rangle$. So we have lost the original state (forever) and have not gained any information because the final state depends on *our own* choice of basis. We will see later in this course, that the quantum analog of Shannon's entropy, the Von Neumann entropy, associated to a state like $|\psi\rangle$ is zero. Note however that if we are given many copies of $|\psi\rangle$ we can measure all of them in the same basis and get a hold of the probabilities $|\langle b_1 | \psi \rangle|^2, |\langle b_2 | \psi \rangle|^2$.

1.6 A random number generator

At this point the reader may well wonder if quantum laws offer any useful resource in order to process information. In this course we will see that this is so. Here we illustrate this with a very simplified model for a random number generator. A corresponding technology working along the lines explained here exists, and is being commercialized.

A source sends a beam of photons on a semi-transparent mirror (figure 1.8). The later splits the beam in two parts, the transmitted and reflected beams. If the source is classical we observe that the two detectors each collect a fraction of the incoming intensity of the beam. Assuming that the semi-transparent mirror is perfect each detector collects half of the intensity.

When the intensity of our source is lowered sufficiently so that it becomes

a single photon source. Photons go through the mirror one at a time, we observe that *either* D_H or D_V clicks, never the two at the same time. We obtain a sequence of clicks 01000111010101000110111100 that looks Bernoulli with parameter $p = \frac{1}{2}$.

The interpretation of this experimental setup, in the framework of quantum mechanics, is as follows. We drop the polarization index as it plays no role here. A single photon is incoming in the semi-transparent mirror and the state of the photon after the mirror is,

$$e^{i\omega t} \frac{1}{\sqrt{2}}(|\mathbf{k}_H\rangle + |\mathbf{k}_V\rangle) \quad (1.50)$$

This state is a superposition. The outcome of the measurement by the detectors cannot be predicted. The probability that the photon is observed in state $|\mathbf{k}_H\rangle$ is

$$|\langle \mathbf{k}_H | e^{i\omega t} \frac{1}{\sqrt{2}}(|\mathbf{k}_H\rangle + |\mathbf{k}_V\rangle)|^2 = \frac{1}{2} \quad (1.51)$$

and similarly the probability that it is observed in state \mathbf{k}_V is

$$|\langle \mathbf{k}_V | e^{i\omega t} \frac{1}{\sqrt{2}}(|\mathbf{k}_H\rangle + |\mathbf{k}_V\rangle)|^2 = \frac{1}{2} \quad (1.52)$$

So the measurement process produces a perfectly random sequence. What do we mean by "perfectly random sequence"? Of course, the sequence is perfectly random only in principle, because in the real experiment there are imperfections, for example, the source is only approximately a single photon source and the semi-transparent mirror has a small bias ect.... But the point here is that, according to the standard interpretation of quantum mechanics, the measurement process produces "true randomness" and not "pseudorandomness": the clicks are not the result of some underlying deterministic process. This point has been much debated by the founding fathers of 20-th century physics and notably by Einstein and Bohr. According to Einstein "God does not play dice", a view that Bohr dismissed. Until today, no other theoretical framework has, succesfully described as many phenomena as quantum theory does, and we have so far no experiment that forces us to abandon the standard quantum framework. It is in this sense that we declare the sequence perfectly random.

A sligthly more abstract representation in quantum information theory language of this experiment is depicted on figure 1.9. We prepare and measure states in the computational basis $|0\rangle, |1\rangle$. The initial state is $|0\rangle$ goes through a Hadamard gate

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (1.53)$$

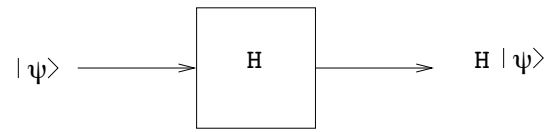


Figure 1.9: Hadamard gate as a model for a semi-transparent mirror

which produces the state

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \quad (1.54)$$

When we perform a measurement on (1.54) the outcome is $|0\rangle$ with probability

$$|\langle 0|H|0\rangle|^2 = \frac{1}{2} \quad (1.55)$$

or $|1\rangle$ with probability

$$|\langle 1|H|0\rangle|^2 = \frac{1}{2} \quad (1.56)$$