Exercises november 27, 2009. Quantum information theory and computation

Exercise 1. Source coding of mixed states: a conjecture as of 2009.

We consider a source of mixed states ρ_x occuring each with probabilities p_x. Messages are N letter strings of the form $\rho_{x_1} \otimes .... \otimes \rho_{x_2}$ and have a probability $p_{x_1}...p_{x_N}$. In this exercise we want to give some support to the conjecture that the achievable rate of compression for a source of mixed states is equal to the Holevo quantity

$$\chi(\{p_x, \rho_x\}) = S(\rho) - \sum_x p_x S(\rho_x), \qquad \rho = \sum_x p_x \rho_x$$

Note that in the case of a source of pure states $\rho_x = |\phi_x\rangle\langle\phi_x|$ the Holevo quantity $\chi(\rho)$ reduce to $S(\rho)$ which is the optimal achievable rate given by Schumacher's theorem.

a) Take a source constituted of the unique letter ρ_0 occuring with probability $p_0 = 1$. How many bits are needed to compress this source ? What is the value of $\chi(\rho)$? Is this consistent ?

b) Now consider a source of mixed mutually orthogonal states. Two mixed states are said to be mutually orthogonal if

$$Tr\rho_x\rho_y = 0, \qquad x \neq y$$

Construct purifications $|\Psi_x\rangle$ of ρ_x that satisfy (*hint: use the spectral decomposition*)

$$\langle\Psi_x|\Psi_y\rangle = 0, \qquad x \neq y$$

What would be an encoding scheme achieving a compression rate of $H(X) = -\sum_x p_x \log p_x$? Why would this rate be optimal ? Check that in the present case we have

$$H(X) = \chi(\rho)$$

hint: no big calculations.

Exercise 2. Product state capacity of the quantum depolarizing channel.

We want to calculate the capacity of the depolarizing channel with noise strength $0 \leq \epsilon \leq 1$. This channel is defined by the map (take the opportunity to check the formulas of the course, although you dont really need them for this exercise)

$$\mathcal{Q}(\rho) = (1-\epsilon)\rho + \frac{\epsilon}{2} I$$

The product state capacity is given by

$$C(\epsilon) = \max_{p_x,\rho_x} \chi(\{p_x, \mathcal{Q}(\rho_x)\})$$

where the supremum is taken over a finite alphabet $\{\rho_x\}$ of 2×2 density matrices and a probability distribution over this alphabet. We will accept (and there is a proof) that the max is attained for pure states $\rho_x = |\phi_x\rangle\langle\phi_x|$ (not necessarily orthogonal).

Prove that

$$C(\epsilon) = \ln 2 - H\big(\frac{\epsilon}{2}\big)$$

were $H(x)$ is the usual binary entropy function (defined with natural log).

hint: no big optimization.