Exercises october 23, 2009. Quantum information theory and computation

Exercise 1. Entanglement swapping

Let O, A,A', B and B' be located at coordinates 0, $-L$, $-\frac{L}{2}$, L and $\frac{L}{2}$ respectively. We suppose that two EPR pairs are produced at A' and B'. For each pair the entangled particles are then propagated to A and O and to B and O. Thus we have an entangled Bell state between A and O and another entangled Bell state between B and O. If the state of the four particles (or four Qbits) is

$$\frac{1}{\sqrt{2}}(|00\rangle_{AO} + |11\rangle_{AO}) \otimes \frac{1}{\sqrt{2}}(|00\rangle_{OB} + |11\rangle_{OB})$$

explain what happens if we make a measurement in the Bell basis of the two Qbits located at O.

Now consider three close by locations A, B, C (for example three points in your lab) and three distant locations A', B', C'. Suppose we have created three entangled pairs between AA', BB', CC' in the state

$$\frac{1}{\sqrt{2}}(|00\rangle_{AA'} + |11\rangle_{AA'}) \otimes \frac{1}{\sqrt{2}}(|00\rangle_{BB'} + |11\rangle_{BB'}) \otimes \frac{1}{\sqrt{2}}(|00\rangle_{CC'} + |11\rangle_{CC'})$$

What happens if we do a measurement in the GHZ basis of the three particles at A, B, C ?

Hint: the GHZ states have been constructed last week, the first state of the 8 dimensional basis of fully entangled states is $\frac{1}{\sqrt{2}}(|000\rangle_{ABC} + |111\rangle_{ABC})$

Exercise 2. GHZ states and "local hidden variable theories"

The goal of this exercise is to discuss a thought experiment that proves that QM results cannot be replaced by local hidden variable theories. Consider a GHZ state of three spins $|GHZ\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\rangle_{ABC} - |\downarrow\downarrow\downarrow\rangle_{ABC})$ where A, B, C are distant locations (which do not communicate). Consider the three observables X, Y, Z represented by the three Pauli matrices (actually we will not use Z so forget about it).

a) Show that $|GHZ\rangle$ is an eigenstate of the operators $Y_A \otimes Y_B \otimes X_C$, $Y_A \otimes X_B \otimes Y_C$, $X_A \otimes Y_B \otimes Y_C$ with eigenvalue 1. Furthermore show that $|GHZ\rangle$ is an eigenstate of $X_A \otimes X_B \otimes X_C$ with eigenvalue -1.

b) Now imagine Alice, Bob and Charlie in their labs at locations A, B and C measure the observables X and Y on their respective particles. They do the four experiments (each time on a new GHZ state):

- experiment one: Alice measures Y, Bob Y and Charlie X.
- experiment two: Alice measures Y, Bob X and Charlie Y.
- experiment three: Alice measures X, Bob Y and Charlie Y.
- experiment four: Alice measures X, Bob X and Charlie X.

Give the resulting states and the associated probability after each experiment according to QM.

c) Suppose now that the outcome of any measurement can be described by a local hidden variable theory. In other words suppose that Alice, Bob and Charlie have some way of computing the outcome of their experiments by some functions $F_A(W, \Lambda)$, $F_B(W, \Lambda)$, $F_C(W, \Lambda)$ where the first variable W is the measurement basis (or apparatus) used i.e $W = X, Y$ and the second variable Λ is the "hidden variable" of the theory (e.g state of the rest of the universe). Show that this setting is not compatible with the QM results of the four previous experiments.

Hint: there is no big calculation, you only have to multiply plus and minus ones ! When the spin is $\uparrow$ record a $+1$ for $F_{A,B,C}(W, \Lambda)$ and when it is $\downarrow$ record a -1 for $F_{A,B,C}(W, \Lambda)$.

Exercise 3. Entanglement purification

The purpose of the exercise is to show from non-fully entangled states we can create with finite probability fully entangled states. We will come back to this later in the course when we will treat "entanglement purification". We have four Qbits in the state $|\Psi_\alpha\rangle \otimes |\Psi_\alpha\rangle$ where

$$|\Psi_\alpha\rangle = \alpha|00\rangle + (1 - \alpha^2)^{1/2}|11\rangle$$

We measure the observable

$$Z \otimes I \otimes I \otimes I + I \otimes I \otimes Z \otimes I$$

Give all the possible outcomes of this measurement together with their respective probabilities. What is the probability that we obtain a fully entangled state ?

Hint: write the observable in the Dirac notation.

Exercise 4. Bell inequality for a non-maximally entangled state.

Calculate the QM prediction for the CHSH quantity (we called it X in the lecture on Bell's inequality) when the EPR pair is produced in the state

$$|\Psi_\alpha\rangle = \alpha|00\rangle + (1-\alpha^2)^{1/2}|11\rangle$$

Repeat the calculations done in the notes to show that the maximal value of X is $2[1+4\alpha^2(1-\alpha^2)^{1/2}]^{1/2}$. In this sense we can say that $\alpha = \sqrt{2}$ corresponds to a maximally entangled state.