Exercises november 16, 2005. Quantum information theory and computation

Exercise 1. Entanglement of Bell states

Prove that the four Bell states belonging to the Hilbert space $\mathbf{C}^2 \otimes \mathbf{C}^2$ are entangled. In other words you have to show that it is not possible to find $|\phi\rangle \in \mathbf{C}^2$ and $|\psi\rangle \in \mathbf{C}^2$ such that a Bell state equals $|\phi\rangle \otimes |\psi\rangle$.

Exercise 2. Three particle entanglement - GHZ states

Find the simplest three Qbit *fully entangled* state you can think of. Here fully entangled means that it cannot be written as the tensor product of three one Qbit states *and* it cannot be written as the tensor product of a two with a one Qbit state *and* it cannot be written as a tensor product of a one with a two Qbit state. Think of a way of generating such a state from tensor product state $|000\rangle$ by a unitary operation and give the corresponding quantum circuit. Find an orthonormal basis of entangled states for the Hilbert space $\mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \mathbf{C}^2$.

The simplest such states are called GHZ states after Greene, Horne and Zeilinger. They can be produced and manipulated experimentally.

Exercise 3. Tsirelson inequality and maximal violation of Bell's inequality

The purpose of the exercise is to show that the set up described in the course yields the maximum possible violation of the Bell inequality.

The three 2×2 matrices X, Y, Z are called Pauli matrices. In the Dirac notation they are $X = |0\rangle\langle 1| + |1\rangle\langle 0|$, $Y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$ and $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$. In physics the standard notation for these matrices is σ_x, σ_y and σ_z.

It is often convenient to introduce the "vector" $\sigma = (X, Y, Z)$. For electrons this has the physical meaning of the "spin of the electron". For photons it simply corresponds to three different polarisation observables: linear (say 45 degrees), circular, linear (say 0 degree).

a) Check the commutation relations $[X, Y] = 2iZ$, $[Y, Z] = 2iX$, $[Z, X] = 2iY$.

b) Let $Q = \mathbf{q} \cdot \sigma$ and $R = \mathbf{r} \cdot \sigma$. Check $[Q, R] = 2i(\mathbf{q} \times \mathbf{r}) \cdot \sigma$

c) Let also $S = \mathbf{q} \cdot \sigma$ and $T = \mathbf{t} \cdot \sigma$. Prove the identity

$$R \otimes S + R \otimes S + R \otimes T - Q \otimes T = 4I + [Q, R] \otimes [S, T]$$

and deduce that for any state $|\psi\rangle$ of $\mathbf{C}^2 \otimes \mathbf{C}^2$ we have the inequality

$$\langle \psi | R \otimes S + R \otimes S + R \otimes T - Q \otimes T | \psi \rangle \leq 2\sqrt{2}$$

d) What are $|\psi\rangle$, $\mathbf{q}$, $\mathbf{r}$, $\mathbf{s}$, $\mathbf{t}$ in the experimental setup of described in the course on the violation of Bell's inequality? What is the general significance of the above inequality ?