Exercises october 2, 2009. Quantum information theory and computation

Exercise 1. Heisenberg uncertainty relation

a) Prove Heisenberg's uncertainty relation (see notes)

$$\Delta A \cdot \Delta B \geq \frac{1}{2} |\langle \psi | [A, B] | \psi \rangle|$$

Hint: consider the commutator of A' and B' where $A' = A - \langle \psi | A | \psi \rangle$ and similarly for B. Use Cauchy- Schwartz !

b) Take $|\psi\rangle = |0\rangle$, $A = X$, $B = Y$ and apply the inequality. If the Qbit is a spin one-half and X, Y, Z are the components of the spin (a kind of "vector"), how do you interpretet the inequality ? (Pauli matrices X, Y, Z are defined in the notes)

c) This question lies a bit outside of the scope of this course but anyone learning QM should be exposed to it. Consider now the Hilbert space $\mathcal{H} = L^2(\mathbf{R})$ of a particle in one dimensional space. The states are wave functions $\psi(x)$ that are square integrable. The position observable is the multiplication operator $\hat{x}$ defined by $(\hat{x}\psi)(x) = x\psi(x)$ and the momentum operator $\hat{p}$ defined by $(\hat{p}\psi)(x) = -i\hbar \frac{d}{dx}\psi(x)$. Compute the commutator $[\hat{x}, \hat{p}]$ and interpret the uncertainty relation.

Exercise 2. Entropic uncertainty principle

Let A and B be two observables with non-degenerate eigenvector basis $\{|a\rangle\}$ and $\{|b\rangle\}$. Consider the probability distributions given by the measurement postulate when the system is in state $|\psi\rangle$ and the corresponding Shannon entropies. Prove the "entropic uncertainty principle" mentioned in the notes:

$$H_A + H_B \geq -2\log\left(\frac{1 + \max|\langle a|b\rangle|}{2}\right)$$

Hint: Reason geometrically to show that $|\langle a|\psi\rangle\langle\psi|b\rangle|^2 \leq |\langle a|b\rangle|^2$

Exercise 3. No-cloning

With the classical controlled NOT (CNOT) gate we can copy a classical bit $b \in \{0, 1\}$. Such a copy machine is implemented by the circuit of figure 1. The quantum CNOT gate is the unitary matrix s.t $U|0,0\rangle = |0,0\rangle$, $U|0,1\rangle = |0,1\rangle$, $U|1,0\rangle = |1,1\rangle$, $U|1,1\rangle = |1,0\rangle$ (you may want to write down this matrix once in the canonical basis). Suppose in the above circuit the input is $\alpha|0\rangle + \beta|1\rangle$ for the first Qbit and $|0\rangle$ for the second Qbit.

a) For which values of α and β does this machine copy the input Qbit ?

b) Consider now an arbitrary, but given, orthonormal basis of $\mathbf{C}^2$. Use the above CNOT gate (for the computational basis) to construct a copy machine that copies the orthonormal basis.

c) Generalize this last construction to an N Qbit system with Hilbert space $\mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \cdots \otimes \mathbf{C}^2$..

Exercise 4. Production of Bell entangled states

a) Show that the four Bell states of two Qbits form an orthonormal basis of the two Qbit Hilbert space.

b) Show that the circuit of figure 2 (or "unitary machine") produces Bell states from tensor product inputs $|x\rangle \otimes |y\rangle$.

c) What is the unitary matrix corresponding to this circuit ? Compute explicitly this matrix in the canonical basis $\{|0,0\rangle, |0,1\rangle, |1,0\rangle, |1,1\rangle\}$.

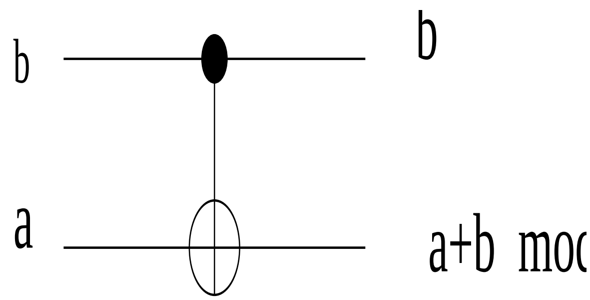

Figure 1: CNOT gate

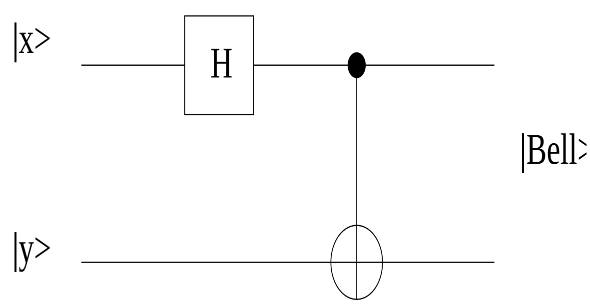

Figure 2: Machine for producing Bell states