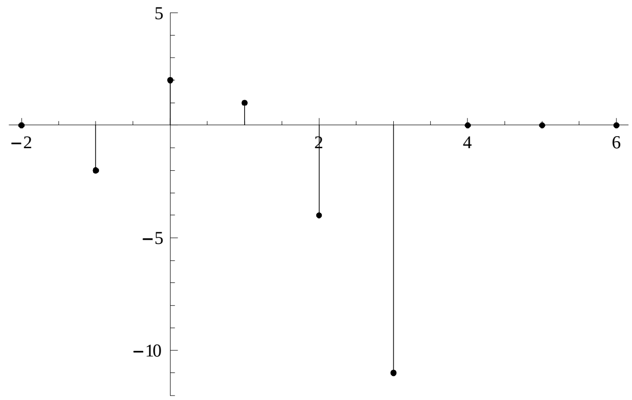
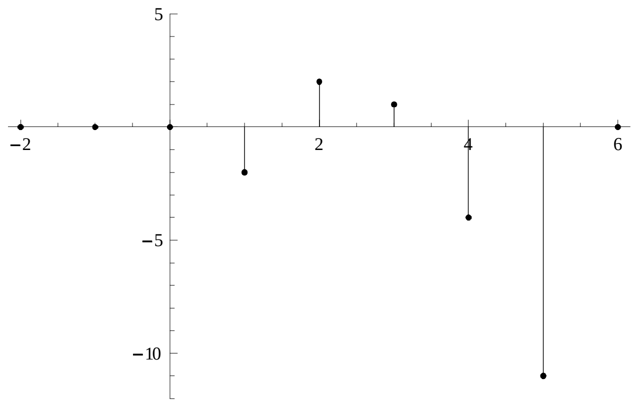
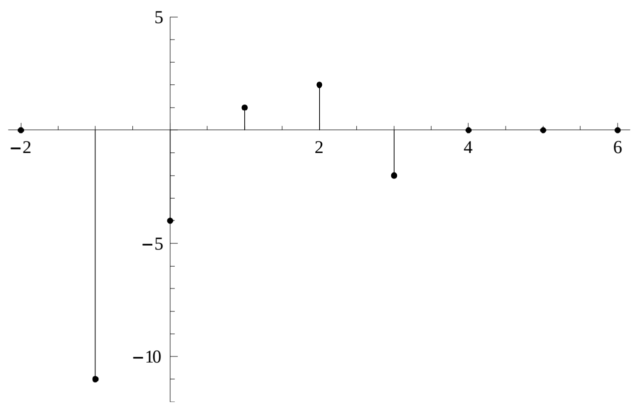
Problem 1



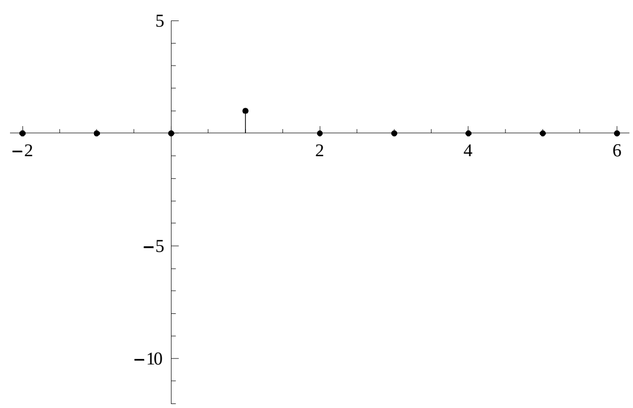
a)



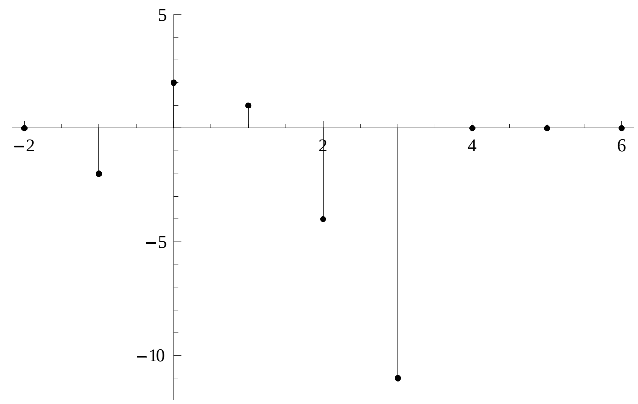
b)



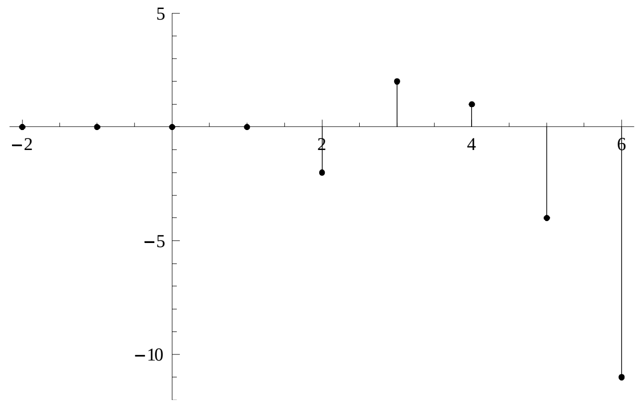
c)



d)

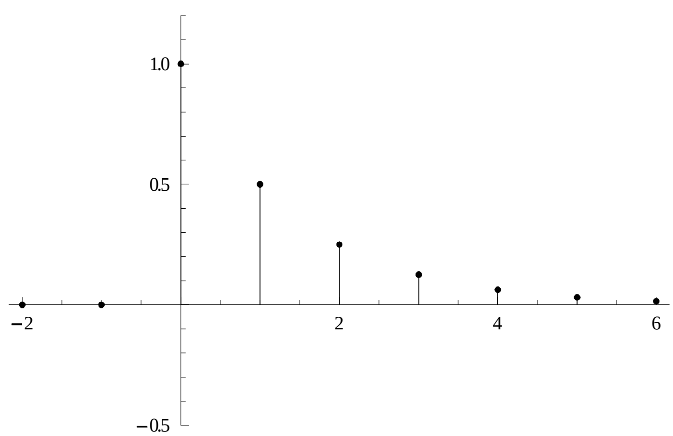


e)



f)

Problem 2



a)

b) The system is causal because $h(n) = 0$ for $n < 0$.

c)

$$\begin{aligned}(1-r)\sum_{k=0}^{n} r^k &= \sum_{k=0}^{n} r^k - r\sum_{k=0}^{n} r^k \\ &= \sum_{k=0}^{n} r^k - \sum_{k=0}^{n} r^{k+1} \\ &= \sum_{k=0}^{n} r^k - \sum_{k=1}^{n+1} r^k \\ &= r^0 + \sum_{k=1}^{n} r^k - \left(r^{n+1} + \sum_{k=1}^{n} r^k\right) \\ &= 1 - r^{n+1}\end{aligned}$$

d) The system is stable because

$$\begin{aligned}\sum_{k=-\infty}^{\infty} |h(k)| &= \sum_{k=0}^{\infty} 2^{-k} \\ &= \lim_{k\to\infty} \frac{1-2^{-(k+1)}}{1-2^{-1}} \\ &= 2 < \infty\end{aligned}$$

e)

$$\begin{aligned}\frac{d}{dr}\frac{1-r^{n+1}}{1-r} &= \frac{d}{dr}\sum_{k=0}^{n} r^k \\ &= \sum_{k=0}^{n} \frac{d}{dr} r^k \\ &= \sum_{k=0}^{n} k r^{k-1} \\ &= r^{-1}\sum_{k=0}^{n} k r^k\end{aligned}$$

f)

$$\begin{aligned}y(n) &= \sum_{k=-\infty}^{\infty} h(k)\, x(n-k) \\ &= \sum_{k=0}^{\infty} 2^{-k}(n-k) \\ &= n\sum_{k=0}^{\infty} 2^{-k} - \sum_{k=0}^{\infty} k\ 2^{-k} \\ &= 2n - 2\frac{d}{dr}\left(\frac{1}{1-r}\right)\Bigg|_{r=2} \\ &= 2n - \frac{1}{2}\left(\frac{1}{(1-1/2)^2}\right) \\ &= 2(n-1)\end{aligned}$$

Problem 3

We have a system from which we only know a linear time-invariant recurence relation between the input $x(n)$ and the output $y(n)$, namely

$$y(n+1) = y(n) + x(n)$$
$$\lim_{m\to-\infty} y(m) = 0$$

a) $y(4) = y(3)+x(3) = y(2)+x(2)+x(3) = y(1)+x(1)+x(2)+x(3) = y(0)+x(0)+x(1)+x(2)+x(3)$

b) Since $\lim_{m\to-\infty} y(m) = 0$

$$\begin{aligned} y(n) &= \lim_{m\to-\infty} y(n) \\ &= \lim_{m\to-\infty} \left(y(m) + \sum_{k=m}^{n-1} x(k) \right) \\ &= \sum_{k=-\infty}^{n-1} x(k) \end{aligned}$$

The output signal is a summation of the input signal.

c) For $n < 0$, since $\delta(n) = 0$ then $h(n+1) = h(n)$ and $h(n+1) = 0$ if $n < 0$. $h(1) = \delta(0) = 1$ and for $n \geq 1$ since $\delta(n) = 0$ then $h(n+1) = h(n) = h(n-1) = ... = h(1) = 1$. We finally found that

$$h(n) = \begin{cases} 1 & \text{if } n \geq 1 \\ 0 & \text{otherwise} \end{cases}.$$

d) The general output is

$$\begin{aligned} y(t) &= \sum_{k=-\infty}^{\infty} h(k)\, x(n-k) \\ &= \sum_{k=1}^{\infty} x(n-k) \\ &= \sum_{k=-\infty}^{-1} x(n+k) \\ &= \sum_{k=-\infty}^{n-1} x(k) \end{aligned}$$

This is same answer than the one found in *b*)

e) The system is unstable because $\sum_{k=\infty}^{\infty} |h(k)| = \sum_{k=1}^{\infty} 1 = \infty$. An example of unstable input signal is $x(n) \equiv 1$.

Problem 4

$$\sin(10\pi t) + \sin(30\pi t)$$

a) The maximum frequency between these two sinusoïds is $15Hz$. So $f_s > 30Hz$.

b) If we use the sampling frequency $f_s = 20Hz$, what the sampled signal would be?

$$\begin{aligned}\sin\left(10\pi\frac{n}{20}\right) + \sin\left(30\pi\frac{n}{20}\right) &= \sin\left(\frac{\pi}{2}n\right) + \sin\left(\frac{3\pi}{2}n\right)\\ &= \sin\left(\frac{\pi}{2}n\right) + \sin\left(-\frac{\pi}{2}n\right)\\ &= \sin\left(\frac{\pi}{2}n\right) - \sin\left(\frac{\pi}{2}n\right)\\ &= 0\end{aligned}$$

c) The ideal filter will supress all frequency greater than $\frac{f_s}{2}$. So the sampled signal would be

$$\sin\left(10\pi\frac{n}{20}\right) = \sin\left(\frac{\pi}{2}n\right)$$

d) The sampling frequency is $f_s = 25Hz$ and we use an ideal interpolator. What the reconstructed signal would be? The first sinusoïd is perfectly reconstructed beacause $5Hz < \frac{25}{2}Hz$, but the second one is altered. An alias frequency for $15Hz$ and lying in the interval $(-25/2, 25/2)$ is given by $15 - 25 = -10Hz$. Thus the reconstructed signal would be

$$\sin(10\pi t) - \sin(20\pi t)$$

Problem 5

We have a continuous signal $x(t)$ with a missing part between $t = 1$ and $t = -1$. The signal is zero everywhere and the only information we know in the missing part is $x(0) = 1$. We want to interpolate this signal.

a) We have to solve two linear system of equation, one for $\{x(-1), x(0)\}$ and an other for $\{x(0), x(1)\}$. For the first part we have

$$\begin{cases} -a + b = 0 \\ b = 1 \end{cases}$$

this gives $a = 1$ and $b = 1$. For the second part we have

$$\begin{cases} a + b = 0 \\ b = 1 \end{cases}$$

and we have $a = -1$ and $b = 1$.

b) We have to solve the following linear system of equation

$$\begin{cases} a - b + c = 0 \\ c = 1 \\ a + b + c = 0 \end{cases}$$

we find $a = -1$, $b = 0$ and $c = 1$.

c) Now we have two equations given by the derivatives i.e. $3at^2 + 2bt + c$. The system is

$$\begin{cases} -a + b - c + d = 0 \\ d = 1 \\ 3a - 2b + c = 0 \\ c = 1 \end{cases}$$

and we find $a = -2$, $b = -3$, $c = 0$, $d = 1$