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Handout 4 Introduction to Communication Systems
Solutions to Homework 2 October 1, 2009

PROBLEM 1.

- Code 1 is not uniquely decodable and not prefix free. For instance if we observe the sequence 011, we do not know if it corresponds to $s_1 s_4$ or $s_1 s_2 s_2$.
- Code 2 is prefix free since no symbol is the prefix of an other one. So it is also uniquely decodable. Also since it is prefix free, it is also instantaneous.
- Code 3 is not prefix free since s_1 is the prefix of s_2 and s_3. However it is uniquely decodeable. Comparing with the prefix free code Code 2, we see that the codewords of Code 3 are the reverse of Code 2's. That Code 2 is uniquely decodeable proves that Code 3 is too, since any string from Code 3 is identical to a string from Code 2 read backwards.
  Hint: Make an example and think for 5 seconds. :D

PROBLEM 2.

- 010011111011111010.
- $b\ l\ a\ b\ l\ a$.
- It is not possible to have a uniquely decodable code for the present source S with the lengths of the codewords restricted to be less than or equal to 2. This is because of the following reason. If we can prove that for any lengths l_a, l_b, l_c, l_d, l_l of the codewords for the symbols a, b, c, d, r respectively, the Kraft's inequality is not satisfied, then there can not exist a uniquely decodable code, since we know that if a code is uniquely decodable then it must satisfy the Kraft's inequality.

  Since any length, $l \leq 2$, we must have that $2^{-l} \geq 2^{-2}$. Thus

  $$2^{-l_a} + 2^{-l_b} + 2^{-l_c} + 2^{-l_d} + 2^{-l_l} \geq 2^{-1} + 2^{-2} + 2^{-2} + 2^{-2} + 2^{-2} > 1$$

  Note that we have one 2^{-1} in the above inequality. This is because we want a uniquely decodable code, hence we cannot have all the codewords of length 2 representing 5 symbols which would immediately violate unicity. Thus we must have at least one codeword of length 1.

  Since for no assignment of the lengths the Kraft's inequality is satisfied we cannot construct a uniquely decodable code.

PROBLEM 3.

- Since
  $$2^{-2} + 2^{-2} + 2^{-2} + 2^{-3} + 2^{-3} = 1$$
  the Kraft's inequality is satisfied.
- Note that Kraft's inequality is a necessary condition. It means that for any uniquely decodable code, the lengths of that code satisfies the Kraft's inequality. The given code satisfies Kraft's inequality, but we *cannot* conclude that it is uniquely decodable. For this we must look at the code in more detail. But by simple inspection of the codewords we see that the code is prefix-free which implies that the code is uniquely decodable.

REMARK 1. *Note that in general if we want to show that the code is uniquely decodable or note, we first look at the Kraft's inequality. If the lenghts* do not satisfy *the Kraft's inequality then for sure the code is* not uniquely decodable. *If it satisfies the Kraft's inequality, then we can see if the code is prefix-free or not. If the code is prefix-free then clearly it is uniquely decodable. If it is not, we then need to look at the codewords in more detail and argue logically that the code is uniquely decodable.*

- The average length of the code is $L = 2.3$.

$$L = 2 * 0.25 + 2 * 0.25 + 2 * 0.2 + 3 * 0.15 + 3 * 0.15 = 2.3.$$

PROBLEM 4.

- Using the hint we see that $\lceil \log_2(\frac{1}{p_i}) \rceil > \log_2(\frac{1}{p_i})$, since $\log_2(\frac{1}{p_i}) \geq 0$. Thus

$$\begin{aligned} \sum_i 2^{-\lceil \log_2(\frac{1}{p_i}) \rceil} &\leq \sum_i 2^{-\log_2(\frac{1}{p_i})} \\ &\leq \sum_i p_i = 1 \end{aligned}$$

Thus Kraft's inequality is satisfied which implies that *there exists a code with lengths given by* $\lceil \log_2(\frac{1}{p_i}) \rceil$ *which is uniquely decodable.*

- The average length of the code is defined as $L_{\text{avg}} = \sum_i p_i l_i$. Here $l_i = \lceil \log_2(\frac{1}{p_i}) \rceil$. Again using the hint we see that $\lceil \log_2(\frac{1}{p_i}) \rceil \leq \log_2(\frac{1}{p_i}) + 1$. Thus we get

$$\begin{aligned} \sum_i p_i \lceil \log_2(\frac{1}{p_i}) \rceil &\leq \sum_i p_i (\log_2(\frac{1}{p_i}) + 1) \\ &= \sum_i p_i (\log_2(\frac{1}{p_i}) + 1 \\ &= H(S) + 1 \end{aligned}$$