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**Handout 20** Advanced Digital Communications
Homework 9 December 14, 2009

PROBLEM 1. (a)

$$\mathbf{y} = \mathbf{Hx} + \mathbf{z}$$

$$\mathbf{Fy} = \mathbf{FSPx} + \mathbf{Fz}$$

$$\Rightarrow \quad \mathbf{Y} = \mathbf{FSF}^*\mathbf{DFx} + \mathbf{Fz}$$

$$\Rightarrow \quad \mathbf{Y} = \underbrace{\mathbf{FSF}^*\mathbf{D}}_{\mathbf{G}}\mathbf{X} + \mathbf{Z}$$

$$\Rightarrow \quad \mathbf{Y} = \mathbf{GX} + \mathbf{Z}$$

(b)

$$\mathbf{Y}_l = \mathbf{G}_{l,l}\mathbf{X}_l + \underbrace{\sum_{q \neq l} \mathbf{G}(l,q)\mathbf{X}_q + \mathbf{Z}_l}_{\text{ICI + noise}}, \quad l = 0, \ldots, N-1,$$

Hence,

$$\text{SINR} = \frac{\mathbb{E}\left(|\mathbf{G}_{l,l}\mathbf{X}_l|^2\right)}{\mathbb{E}\left(\left|\sum_{q\neq l}\mathbf{G}_{l,q}\mathbf{X}_q\right|^2\right) + \mathbb{E}\,|Z_l|^2} = \frac{\mathcal{E}_x|\mathbf{G}_{l,l}|^2}{\mathcal{E}_x\sum_{q\neq l}|\mathbf{G}_{l,q}|^2 + \sigma_z^2}$$

(c)

$$\begin{aligned} \mathbb{E}(\mathbf{YY}^*) &= \mathbb{E}((\mathbf{GX}+\mathbf{Z})(\mathbf{X}^*\mathbf{G}^* + \mathbf{Z}^*)) \\ &= \mathcal{E}_x\mathbf{GG}^* + \mathbf{I}\sigma_z^2. \end{aligned} \tag{1}$$

$$\begin{aligned} \mathbb{E}(\mathbf{X}_l\mathbf{Y}^*) &= \mathbb{E}(\mathbf{X}_l(\mathbf{X}^*\mathbf{G}^* + \mathbf{Z}^*)) \\ &= {}_l^T\mathcal{E}_x\mathbf{G}^*, \end{aligned} \tag{2}$$

where ${}_l^T = \begin{bmatrix} 0 & \cdots & \underbrace{1}_{l^{\text{th}}\text{position}} & 0 & \cdots & 0 \end{bmatrix}$. Orthogonality principle implies,

$$\mathbb{E}((\mathbf{W}_l^*\mathbf{Y} - \mathbf{X}_l)\mathbf{Y}^*) = 0$$

$$\Rightarrow \mathbb{E}(\mathbf{W}_l^*\mathbf{YY}^*) = \mathbb{E}(\mathbf{X}_l\mathbf{Y}^*)$$

$$\Rightarrow \mathbf{W}_l^* = \mathbb{E}(\mathbf{X}_l\mathbf{Y}^*)(\mathbb{E}(\mathbf{YY}^*))^{-1}$$

Using equations 1,2 we get that,

$$\Rightarrow \mathbf{W}_l^* = \mathcal{E}_{x}{}_l^T\mathbf{G}^*(\mathcal{E}_x\mathbf{GG}^* + \mathbf{I}\sigma_z^2)^{-1}$$

(d)

$$\mathbf{G}_{l,q} = (\mathbf{FS})_l(\mathbf{F}^*\mathbf{D})_q$$

where $(\mathbf{FS})_l$ denotes the $l^{th}$ row of $\mathbf{FS}$ and $(\mathbf{F}^*\mathbf{D})_q$ denotes the $q^{th}$ column of $\mathbf{F}^*\mathbf{D}$.

$$\mathbf{G}_{l,q} = \frac{1}{N}\left[e^{j2\pi f_0(N-1)} \quad e^{j2\pi f_0(N-2)}e^{-j\frac{2\pi}{N}(l-1)} \ldots \quad e^{j2\pi f_0(N-N)}e^{-j\frac{2\pi}{N}(l-1)(N-1)}\right]\begin{bmatrix} d_q \\ d_q e^{j\frac{2\pi}{N}(q-1)} \\ \vdots \\ d_q e^{j\frac{2\pi}{N}(q-1)(N-1)} \end{bmatrix}$$

$$\Rightarrow \quad \mathbf{G}_{l,q} = \frac{d_q}{N}e^{j2\pi f_0(N-1)}\sum_{p=1}^{N} e^{\left(j\frac{2\pi}{N}(q-l)-j2\pi f_0\right)(p-1)}$$

By using the summation formula for the geometric series we get,

$$\mathbf{G}_{l,q} = \frac{d_q}{N}e^{j2\pi f_0(N-1)}\left[\frac{1-e^{-j2\pi f_0 N}}{1-e^{j\frac{2\pi}{N}(q-l-f_0N)}}\right] \quad \text{for} \quad f_0 \neq 0.$$

The ICI is significant when $\mathbf{G}_{l,q}$ is comparable to $\mathbf{G}_{l,l}$. When $f_0N$ is large then this could occur, i.e., there is significant time variation over the block.

PROBLEM 2. (a) The rate is $\frac{N}{N+\nu}$.

(b) Let $\mathbf{B}$ is a filter of length $M$, *i.e.*, only $M$ input symbols contribute in $\mathbf{u}_k$:

$$\mathbf{u}_k = \mathbf{B}\tilde{\mathbf{x}}_M = \begin{bmatrix} b_0 & b_1 \cdots & b_{M-1} \end{bmatrix}\begin{bmatrix} x_k \\ x_{k-1} \\ \vdots \\ x_{k-M+1} \end{bmatrix}.$$

Considering $\mathbf{x}_k$ and $\mathbf{u}_k$ in the same linear space.
We can use the following orthogonality principle:

$$\begin{aligned} \mathbb{E}[(\mathbf{r}_k - \mathbf{u}_k)\mathbf{y}_k^*] &= 0 \\ \mathbb{E}[\mathbf{r}_k\mathbf{y}_k^*] - \mathbb{E}[\mathbf{u}_k\mathbf{y}_k^*] &= 0 \\ \mathbf{W}\mathbb{E}[(\mathbf{H}\mathbf{x}_k + \mathbf{z}_k)(\mathbf{x}_k^*\mathbf{H}^* + \mathbf{z}_k^*)] - \mathbb{E}[\mathbf{B}\tilde{\mathbf{x}}_M(\mathbf{x}_k^*\mathbf{H}^* + \mathbf{z}_k^*)] &= 0 \\ \mathbf{W}\mathbf{H}\mathcal{E}_x\mathbf{I}_{N+\nu}\mathbf{H}^* + \mathbf{W}N_0\mathbf{I}_N - \mathbf{B}\mathcal{E}_x\mathbf{J}\mathbf{H}^* &= 0 \end{aligned}$$

where

$$\mathbf{J} = [\mathbf{I}_M|\mathbf{O}_{M\times N-M+\nu}] = \left[\begin{array}{cccc|ccc} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \end{array}\right].$$

Therefore, we have

$$\mathbf{W} = \mathbf{B}\mathbf{J}\mathbf{H}^*(\mathbf{H}\mathbf{H}^* + \frac{N_0}{\mathcal{E}_x}\mathbf{I}_N)^{-1}$$

(c) In the second channel, we use only a small number ($M$) of received symbols and so the other terms in the convolution product $\sum_{n=0}^{N-1} w_n y_{k-n}$ are considered as a part of noise. Therefore, we have a small decrement in the $SNR$ and and so a little performance loss. Instead of the loss in performance, the rate of the new system is $\frac{N}{N+M}$, which is much more than the rate we have seen in (a). Having lower decoding complexity is the other advantage of the new system.

For reading more about this problem and target channel and its performance, see I. Lee, J. S. Chow, and J. M. Cioffi, "Performance Evaluation of a Fast Computation Algorithm for the DMT in High-Speed Subscriber Loop," *IEEE J. Select. Areas Commun.*, vol. 13, pp. 1564-1570, Dec. 1995.

Problem 3. (a) $\mathbf{y}_k = \sum_{n=0}^{\nu} \mathbf{p}_n x_{k-n} + \mathbf{z}_k$

$$\mathcal{Y}_k = \begin{bmatrix} \mathbf{y}_k \\ \vdots \\ \mathbf{y}_{k-N+1} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{p}_0 & \cdots & \mathbf{p}_\nu & \mathbf{0} & \cdots & \mathbf{0} \\ & \ddots & & \ddots & & \\ & & & \mathbf{p}_0 & \cdots & \mathbf{p}_\nu \end{bmatrix}}_{\mathbf{P}} \underbrace{\begin{bmatrix} x_k \\ \vdots \\ x_{k-N+1-\nu} \end{bmatrix}}_{\mathcal{X}_k} + \underbrace{\begin{bmatrix} \mathbf{z}_k \\ \vdots \\ \mathbf{z}_{k-N+1} \end{bmatrix}}_{\mathcal{Z}_k}$$

(b) Cyclic prefix:

$$x_{k-N-\ell} = x_{k-\ell} \quad , \quad \ell = 0, \ldots, \nu - 1$$

$$\begin{aligned} \mathbf{y}_k &= \mathbf{p}_0 x_k + \cdots + \mathbf{p}_\nu x_{k-\nu} \\ &\vdots \\ \mathbf{y}_{k-N+1+\nu} &= \mathbf{p}_0 x_{k-N+1+\nu} + \cdots + \mathbf{p}_\nu x_{k-N+1} \\ \mathbf{y}_{k-N+\nu} &= \mathbf{p}_0 x_{k-N+\nu} + \cdots + \mathbf{p}_{\nu-1} x_{k-N+1} + \mathbf{p}_\nu \underbrace{x_{k-N}}_{x_k} \\ &\vdots \\ \mathbf{y}_{k-N+1} &= \mathbf{p}_0 x_{k-N+1} + \mathbf{p}_1 \underbrace{x_{k-N}}_{x_k} + \cdots + \mathbf{p}_\nu \underbrace{x_{k-N+1-\nu}}_{x_{k-\nu+1}} \end{aligned}$$

Hence

$$\mathcal{Y}_k = \begin{bmatrix} \mathbf{y}_k \\ \vdots \\ \mathbf{y}_{k-N+1+\nu} \\ \mathbf{y}_{k-N+\nu} \\ \vdots \\ \mathbf{y}_{k-N+1} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{p}_0 & \cdots & \mathbf{p}_\nu & \mathbf{0} & \cdots\cdots & \mathbf{0} \\ & \ddots & & & \ddots & \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{p}_0 & \cdots\cdots & \mathbf{p}_\nu \\ \mathbf{p}_\nu & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{p}_0 \quad \cdots & \mathbf{p}_{\nu-1} \\ & \ddots & & & \ddots & \\ \mathbf{p}_1 & \cdots & \mathbf{p}_\nu & \mathbf{0} & \cdots \quad \mathbf{0} & \mathbf{p}_0 \end{bmatrix}}_{\widetilde{\mathbf{P}}} \underbrace{\begin{bmatrix} x_k \\ \vdots \\ x_{k-N+1} \end{bmatrix}}_{\tilde{\mathcal{X}}_k} + \underbrace{\begin{bmatrix} \mathbf{z}_k \\ \vdots \\ \mathbf{z}_{k-N+1} \end{bmatrix}}_{\mathcal{Z}_k}$$

(c) Here the main point was to observe that although $\widetilde{\mathbf{P}}$ is *not* a circulant matrix, it can be decomposed into block circulant matrix and this is the crucial observation we were looking for. Another approach is to go back to basics and look at equivalent periodic sequences and that argument follows almost the same steps as done in the first derivation of OFDM in class. Therefore we use the block circulant argument in these solutions.

Let

$$\mathbf{p}_n = \begin{bmatrix} \mathbf{p}_n(1) \\ \mathbf{p}_n(2) \end{bmatrix} \in \mathbb{C}^n \quad , \quad \mathbf{y}_k = \begin{bmatrix} y_k(1) \\ y_k(2) \end{bmatrix} \in \mathbb{C}^2$$

Clearly for $q \in \{1, 2\}$,

$$\mathcal{Y}_k(q) \triangleq \begin{bmatrix} y_k(q) \\ \vdots \\ y_{k-N+1+\nu}(q) \\ y_{k-N+\nu}(q) \\ \vdots \\ y_{k-N+1}(q) \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{p}_0(q) & \dots & \mathbf{p}_\nu(q) & \mathbf{0} & \dots\dots\dots & \mathbf{0} \\ & \ddots & & & \ddots & \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{p}_0(q) & \dots\dots\dots & \mathbf{p}_\nu(q) \\ \mathbf{p}_\nu(q) & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{p}_0(q) \quad \cdots & \mathbf{p}_{\nu-1}(q) \\ & \ddots & & & \ddots & \\ \mathbf{p}_1(q) & \cdots & \mathbf{p}_\nu(q) & \mathbf{0} & \cdots \quad \mathbf{0} & \mathbf{p}_0(q) \end{bmatrix}}_{\widetilde{\mathbf{P}}(q)} \underbrace{\begin{bmatrix} x_k \\ \vdots \\ x_{k-N+1} \end{bmatrix}}_{\tilde{\mathcal{X}}_k} + \underbrace{\begin{bmatrix} z_k \\ \vdots \\ z_{k-N+1} \end{bmatrix}}_{\mathcal{Z}_k}$$

and clearly $\widetilde{\mathbf{P}}(q) \in \mathbb{C}^{N \times N}$ is a circulant matrix and therefore has the Fourier basis as its eigen basis if

$$\widetilde{\mathbf{P}}(q) = \mathbf{F}^* \mathbf{D}_q \mathbf{F}$$

where $\mathbf{F}$ is the Fourier transform matrix, the using,

$$\mathbf{X} = \mathbf{F}\mathcal{X}_k, \quad \mathbf{Y}_q = \mathbf{F}\mathcal{Y}_k(q), \quad \mathbf{Z}_q = \mathbf{F}\mathcal{Z}_k(q), \quad q \in \{1, 2\}$$

we get,

$$\mathcal{Y}_k(q) = \mathbf{F}^* \mathbf{D}_q \mathbf{F} \mathcal{X}_k + \mathcal{Z}_k(q)$$

$$\mathbf{Y}_q = \mathbf{F}\mathcal{Y}_k(q) = \mathbf{D}_q \mathbf{X} + \mathbf{Z}_q, \quad q \in \{1, 2\}$$

and then

$$\mathbf{Y}_q(\ell) = \mathbf{D}_q(\ell) X(\ell) + \mathbf{Z}_q(\ell), \quad \ell = 0, \dots, N-1$$

where

$$\mathbf{D}_q(\ell) = \sum_{n=0}^{\nu} p_n(q) \mathrm{e}^{-j\frac{2\pi}{N}\ell n}, \quad \ell = 0, \dots, N-1$$

Therefore

$$\mathbf{Y}(\ell) = \begin{bmatrix} Y_1(\ell) \\ Y_2(\ell) \end{bmatrix} = \underbrace{\left( \sum_{n=0}^{\nu} \begin{bmatrix} p_n(1) \\ p_n(2) \end{bmatrix} \mathrm{e}^{-j\frac{2\pi}{N}\ell n} \right)}_{\mathbf{P}(\ell)} X(\ell) + \underbrace{\begin{bmatrix} Z_1(\ell) \\ Z_2(\ell) \end{bmatrix}}_{\mathbf{Z}(\ell)} \tag{3}$$

(d) One main thing we wanted you to notice was that (3) is a *vector* equation and therefore the MMSE estimator involves matricies and vectors.

$$\widehat{X}(\ell) = \mathbf{W}(\ell)\mathbf{Y}(\ell), \quad \text{where } \mathbf{W}(\ell) \in \mathbb{C}^{1 \times L}, \text{ for } L = 2$$

we want

$$\mathbf{W}(\ell) = \arg\min_{\mathbf{W}(\ell)} \mathbb{E}||X(\ell) - \mathbf{W}(\ell)\mathbf{Y}(\ell)||^2$$

Using the orthogonality principle we get

$$\mathbb{E}[X(\ell) - \mathbf{W}(\ell)\mathbf{Y}(\ell)]\mathbf{Y}^*(\ell) = 0$$

$$\begin{aligned} \mathbb{E}X(\ell)\mathbf{Y}^*(\ell) &= W(\ell)\mathbb{E}\mathbf{Y}(\ell)\mathbf{Y}^*(\ell) \\ \mathbf{W}(\ell) &= \mathbb{E}[X(\ell)\mathbf{Y}^*(\ell)]\left\{\mathbb{E}\mathbf{Y}(\ell)\mathbf{Y}^*(\ell)\right\}^{-1} \end{aligned}$$

Now,

$$\begin{aligned} \mathbb{E}X(\ell)\mathbf{Y}^*(\ell) &= \mathcal{E}_x(\ell)\mathbf{P}^*(\ell) \\ \mathbb{E}\mathbf{Y}(\ell)\mathbf{Y}^*(\ell) &= \mathcal{E}_x(\ell)\mathbf{P}(\ell)\mathbf{P}^*(\ell) + \sigma^2\mathbf{I}_2 \end{aligned}$$

where

$$\begin{aligned} \mathbf{P}(\ell) &= \sum_{n=0}^{\nu} p_n(q)\mathrm{e}^{-j\frac{2\pi}{N}\ell n}, \quad \ell = 0, \ldots, N-1 \\ \mathcal{E}_x(\ell) &= \mathbb{E}|X(\ell)|^2 \end{aligned}$$

Hence

$$\mathbf{W}(\ell) = \mathcal{E}_x(\ell)\mathbf{P}^*(\ell)\left[\mathcal{E}_x(\ell)\mathbf{P}(\ell)\mathbf{P}^*(\ell) + \sigma^2\mathbf{I}_2\right]^{-1}$$

Using the matrix inversion lemma (Woodbury's identity),

$$\begin{aligned} \mathbf{W}(\ell) &= \mathbf{P}^*(\ell)\left[\mathcal{E}_x(\ell)\mathbf{P}(\ell)\mathbf{P}^*(\ell) + \frac{\sigma^2}{\mathcal{E}_x(\ell)}\mathbf{I}_2\right]^{-1} \\ &= \mathbf{P}^*(\ell)\left\{\frac{\mathcal{E}_x(\ell)}{\sigma^2}\mathbf{I}_2 - \left(\frac{\mathcal{E}_x(\ell)}{\sigma^2}\right)\mathbf{P}(\ell)\left[1 + \mathbf{P}^*(\ell)\frac{\mathcal{E}_x(\ell)}{\sigma^2}\mathbf{P}(\ell)\right]^{-1}\mathbf{P}^*(\ell)\frac{\mathcal{E}_x(\ell)}{\sigma^2}\right\} \\ &= \frac{\mathcal{E}_x(\ell)}{\sigma^2}\mathbf{P}^*(\ell)\left\{\mathbf{I}_2 - \frac{\mathbf{P}(\ell)\mathbf{P}^*(\ell)\frac{\mathcal{E}_x(\ell)}{\sigma^2}}{1 + ||\mathbf{P}(\ell)||^2\frac{\mathcal{E}_x(\ell)}{\sigma^2}}\right\} \\ &= \frac{\mathcal{E}_x(\ell)}{\sigma^2}\left\{1 - \frac{||\mathbf{P}(\ell)||^2\frac{\mathcal{E}_x(\ell)}{\sigma^2}}{1 + ||\mathbf{P}(\ell)||^2\frac{\mathcal{E}_x(\ell)}{\sigma^2}}\right\}\mathbf{P}^*(\ell) \\ &= \frac{\mathcal{E}_x(\ell)}{\sigma^2}\frac{1}{1 + ||\mathbf{P}(\ell)||^2\frac{\mathcal{E}_x(\ell)}{\sigma^2}}\mathbf{P}^*(\ell) \end{aligned}$$

(e) Again the main things to notice here is that (3) is a *vector* relationship, and hence the rate $R_\ell$ on each parallel channel is

$$\begin{aligned} R_\ell &= \frac{1}{2}\log\left|\mathbf{I}_2 + \mathbf{P}(\ell)\mathbf{P}^*(\ell)\frac{\mathcal{E}_x(\ell)}{\sigma^2}\right| \\ &= \frac{1}{2}\log\left(1 + ||\mathbf{P}(\ell)||^2\frac{\mathcal{E}_x(\ell)}{\sigma^2}\right) \end{aligned}$$

by using the identity $\det(\mathbf{I}+\mathbf{AB}) = \det(\mathbf{I}+\mathbf{BA})$. Hence the total achievable rate is,

$$R = \sum_{\ell=0}^{N-1} R_\ell = \frac{1}{2}\sum_{\ell=0}^{N-1}\log\left(1 + ||\mathbf{P}(\ell)||^2\frac{\mathcal{E}_x(\ell)}{\sigma^2}\right)$$

and the optimization problem is

$$\max \frac{1}{2} \sum_{\ell=0}^{N-1} \log \left( 1 + ||\mathbf{P}(\ell)||^2 \frac{\mathcal{E}_x(\ell)}{\sigma^2} \right)$$
$$\text{s.t.} \sum_{\ell=0}^{N-1} \mathcal{E}_x(\ell) \leq N P_{\text{tot}}$$

and this has the same forms as the problem studied in class with $|P(\ell)|^2$ replaced by $||\mathbf{P}(\ell)||^2$, and applying the waterfilling algorithm.

PROBLEM 4. 1.

$$\mathbf{P} = \begin{bmatrix} 1 & 1.81 & 0.81 & 0 \\ 0 & 1 & 1.81 & 0.81 \\ 0.81 & 0 & 1 & 1.81 \\ 1.81 & 0.81 & 0 & 1 \end{bmatrix}$$

Let's compute the eigen decomposition:
$\mathbf{F} \in \mathbb{C}^{n \times n}$ and $\mathbf{\Lambda} \in \mathbb{C}^{n \times n}$

$$\mathbf{F} = \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & -0.5i & -0.5 & 0.5i \\ 0.5 & -0.5 & 0.5 & -0.5 \\ 0.5 & 0.5i & -0.5 & -0.5i \end{bmatrix}$$

and

$$\mathbf{\Lambda} = \begin{bmatrix} 3.62 & 0 & 0 & 0 \\ 0 & 0.19 - 1.81i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.19 + 1.81i \end{bmatrix}$$

2. After applying FEQ, we find:

$$SNR = \frac{|d_l|_x^2}{\sigma^2}$$
$$SNR = \{131.044, 33.122, 0, 33.122\}$$