École Polytechnique Fédérale de Lausanne

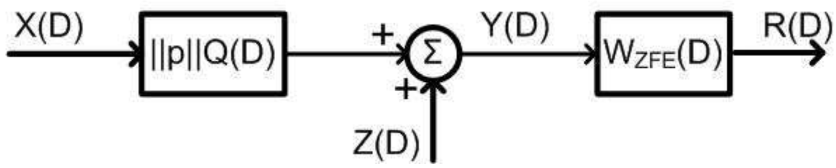
School of Computer and Communication Sciences

**Handout 22** Advanced Digital Communications

Solution 8 January 2, 2010

Problem 1.

(a) (i) Block diagram for ZFE:




(ii)

$$Q(D) = bD + bD^{-1} + 1 = \frac{1}{2}D + 1 + \frac{1}{2}D^{-1} \quad (1)$$

$$W_{ZFE}(D) = \frac{1}{||p||Q(D)} = \frac{1}{\frac{1}{2}||p||(1+D)(1+D^{-1})} \quad (2)$$

$W_{ZFE}(D)$ has a double pole at $-1$, thus it is not a stable filter.

(iii)

$$\overline{\sigma}^2_{ZFE} = \frac{N_0}{2||p||}\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{\frac{1}{2}||p||(1+e^{j\omega})(1+e^{-j\omega})}d\omega \quad (3)$$

$$= \frac{N_0}{2\pi||p||^2}\int_{-\pi}^{\pi}\frac{1}{2+2\cos\omega}d\omega = \infty \quad (4)$$

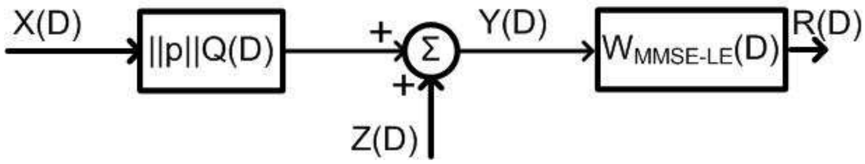
This result is expected because of the unstable behaviour of the filter.

(iv)

$$SNR_{ZFE} = \frac{E_x}{\overline{\sigma}^2_{ZFE}} = 0 \quad (5)$$

This result tells us that the signal is comletely corrupted by noise at the output, and that this equalizer is not suitable for detection.

(b) (i) Block diagram for MMSE-LE:




(ii)

$$W_{MMSE-LE}(D) = \frac{1}{||p||}\frac{1}{Q(D)+b^2} = \frac{1}{||p||}\frac{1}{\frac{1}{2}D+\frac{5}{4}+\frac{1}{2}D^{-1}} \quad (6)$$

$$= \frac{1}{||p||}\frac{4}{2D+5+2D^{-1}} \quad (7)$$

$$= \frac{1}{||p||(1+\frac{1}{2}D)(1+\frac{1}{2}D^{-1})} \quad (8)$$

(iii)

$$R(D) = W_{MMSE-LE}(D)Y(D) = (1-V(D))X(D) + Z'(D) \quad (9)$$

$$W_{MMSE-LE}(D)Y(D) = W_{MMSE-LE}(D)||p||Q(D)X(D) + Z'(D) \quad (10)$$

$$\Longrightarrow 1 - V(D) = ||p||Q(D)W_{MMSE-LE}(D) \quad (11)$$

$$= \Big(\frac{4}{||p||(2D+5+2D^{-1})}\Big)||p||(\frac{1}{2}D+1+\frac{1}{2}D^{-1}) \quad (12)$$

$$= \frac{2D+4+2D^{-1}}{2D+5+2D^{-1}} \quad (13)$$

$$= 1 - \frac{1}{2D+5+2D^{-1}} \quad (14)$$

$$\Longrightarrow V(D) = \frac{1}{2D+5+2D^{-1}} \quad (15)$$

Expanding $V(D)$ into its partial fractions we obtain:

$$V(D) = \frac{1}{(1+2D)(1+2D^{-1})} = \underbrace{\frac{\frac{2}{3}}{D+2}}_{\text{causal}} - \underbrace{\frac{\frac{1}{3}}{2D+1}}_{anti-causal} \quad (16)$$

Taking the inverse D-transform we calculate $v[k]$

$$v[k] = \frac{1}{3}\Big((-\frac{1}{2})^k u[k] + (-2)^k u[-k-1]\Big) \quad (17)$$

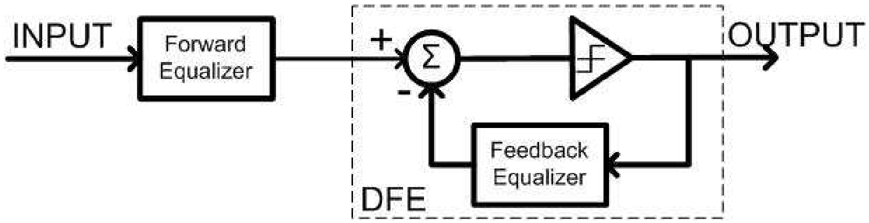
So that $ROC : \frac{1}{2} < |D| < 2$ (contains the unit circle)

(iv) Having computed $v_0 = \frac{1}{3}$ from the previous part;

$$SNR_{MMSE-LE,U} = \frac{1}{v_0} - 1 = 2 \quad (18)$$

Obviously, the MMSE-LE shows better performance than ZFE, having a higher SNR.

(c) (i) Block diagram for MMSE-DFE:




(ii)

$$Q(D) + \frac{1}{SNR_{MFB}} = \frac{1}{2}D + 1 + \frac{1}{2}D^{-1} + \frac{1}{4} = \frac{1}{2}D + \frac{5}{4} + \frac{1}{2}D^{-1} \quad (19)$$

This can be factorized as:

$$Q(D) + \frac{1}{SNR_{MFB}} = (1 + \frac{1}{2}D)(1 + \frac{1}{2}D^{-1}) \quad (20)$$

Giving us the causal monic function $G(D) = 1 + \frac{1}{2}D$ with $\gamma_0 = 1$

(iii)

$$B_{opt}(D) = G(D) = 1 + \frac{1}{2}D \quad (21)$$

(iv)

$$W_{opt}(D) = W_{MMSE-LE}(D)B_{opt}(D) = \frac{1 + \frac{1}{2}D}{||p||(1 + \frac{1}{2}D)(1 + \frac{1}{2}D^{-1})} \quad (22)$$

$$= \frac{1}{||p||(1 + \frac{1}{2}D^{-1})} \quad (23)$$

(v) Using the $G^*(D^{-*})$ and $\gamma_0$ we found before, we define

$$V(D) = \frac{1}{4(1 + \frac{1}{2}D^{-1})} \quad (24)$$

(vi)

$$v_0 = \frac{1}{\gamma_0 SNR_{MFB}} = \frac{1}{4} \quad (25)$$

(vii)

$$SNR_{MMSE-DFE,U} = \frac{1}{v_0} - 1 = 3 \quad (26)$$

(viii) There is an increase in SNR (better equalization) when the decision error feedback equalization is compared to the linear equalization.

Problem 2. (a) From Problem 2 of HW7 we have:

$$\begin{aligned}
||p||^2 &= 1 + aa^* && (27)\\
b &= ||p||^2(1 + \frac{1}{SNR_{MFB}}) && (28)\\
r_2 &= \frac{-b + \sqrt{b^2 - 4aa^*}}{2a} && (29)\\
W_{MMSE-LE}(D) &= \frac{||p||}{a^*D^{-1} + b + aD} = \frac{||p||}{a}\frac{D}{(D-r_1)(D-r_2)} && (30)\\
Q(D) + \frac{1}{SNR_{MFB}} &= \frac{a^*D^{-1} + ||p||^2 + aD}{||p||^2} + \frac{1}{SNR_{MFB}} && (31)\\
&= \frac{a^*D^{-1} + aD}{||p||^2} + 1 + \frac{1}{SNR_{MFB}} && (32)\\
&= \frac{1}{||p||^2}\left(a^*D^{-1} + b + aD\right) = \frac{1}{||p||W_{MMSE-LE}(D)} && (33)\\
&= \frac{a}{||p||^2}\frac{(D-r_1)(D-r_2)}{D} && (34)\\
&= \frac{a}{||p||^2}(D-r_1)(1-r_2D^{-1}) && (35)\\
&= \frac{ar_1}{||p||^2}(Dr_2^*-1)(1-r_2D^{-1}) && (36)\\
&= -\frac{ar_1}{||p||^2}(1-r_2D^{-1})(1-r_2^*D) \quad (since \quad r_1r_2^*=1) && (37)
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow \gamma_0 &= -\frac{ar_1}{||p||^2} && (38)\\
&= \frac{a}{||p||^2}\frac{b + \sqrt{b^2 - 4aa^*}}{2a} && (39)\\
&= \frac{b + \sqrt{b^2 - 4aa^*}}{2(1+aa^*)} && (40)
\end{aligned}$$

(b)

$$\begin{aligned}
B(D) &= G(D) = 1 - r_2^*D && (41)\\
&= 1 - \frac{-b + \sqrt{b^2 - 4aa^*}}{2a^*}D && (42)\\
W(D) &= B(D)W_{MMSE-LE}(D) && (43)\\
&= \frac{B(D)}{||p||\left(Q(D) + \frac{1}{SNR_{MFB}}\right)} && (44)\\
&= -\frac{||p||}{ar_1(1-r_2D^{-1})} && (45)
\end{aligned}$$

(c)

$$SNR_{MMSE-DFE} = \gamma_0 SNR_{MFB} \tag{46}$$

$$\Longrightarrow \gamma_{MMSE-DFE} = 10log_{10}\frac{1}{\gamma_0} \tag{47}$$

$$= 10log_{10}\Big(\frac{2(1+aa^*)}{b+\sqrt{b^2-4aa^*}}\Big) \tag{48}$$

$$where, \quad b = (1+aa^*)\big(1+\frac{1}{SNR_{MFB}}\big) \tag{49}$$

$$= (1+aa^*)\big(1+\frac{N_0}{E_x(1+aa^*)}\big) \tag{50}$$

$$= (1+aa^*)+0.1 = 1.1+aa^* \tag{51}$$

For $a = 0$, $\gamma = 10log_{10}(\frac{1}{1.1}) = -10log_{10}(1.1) = -0.41$

For $a = 0.5$, $\gamma = 10log_{10}(1.11) = 0.44$

For $a = 1$, $\gamma = 10log_{10}(1.46) = 1.64$

The plot for $\gamma_{DFE}$ :