PROBLEM 1.

(a) Via orthogonality, we can write

$$E[(U(D) - \hat{U}(D))Y^*(D^{-*})] = 0$$

where, $U(D) = H(D)X(D)$ and $\hat{U}(D) = W(D)Y(D)$

Thus,

$$E[(H(D)X(D) - W(D)Y(D))Y^*(D^{-*})] = 0 \quad (1)$$

$$E[(H(D)X(D)Y^*(D^{-*})] = E[W(D)Y(D))Y^*(D^{-*})] \quad (2)$$

Using $Y^*(D^{-*}) = ||p||Q^*(D^{-*})X^*(D^{-*}) + Z^*(D^{-*})$, for the R.H.S. we get :

$$\begin{aligned} E[(H(D)X(D)Y^*(D^{-*})] &= H(D)||p||Q^*(D^{-*})E_x + \underbrace{E[H(D)X(D)Z^*(D^{-*})]}_{0} \\ &= ||p||H(D)Q(D)E_x \quad (3) \end{aligned}$$

Since, X_k and Z_k are independent and $Q(D) = Q^*(D^{-*})$

and for the L.H.S. :

$$E[W(D)Y(D))Y^*(D^{-*})] = W(D)[||p||Q^*(D^{-*})E[Y(D)X^*(D^{-*})] + E[Y(D)Z^*(D^{-*}))]]$$

$$= W(D)(||p||Q(D)(E[||p||Q(D)X(D)] + \underbrace{E[Z(D)X^*(D^{-*})])}_{0}] + E[(||p||Q(D)X(D) + Z(D))Z^*(D^{-*})])$$

$$= W(D)(||p||^2Q^2(D)E_x + E[(Z(D)Z^*(D^{-*})] + \underbrace{E[||p||Q(D)X(D)Z^*(D^{-*})]}_{0}$$

$$= W(D)(||p||^2Q^2(D)E_x + N_0Q(D))$$

Thus,

$$W(D)(||p||^2Q^2(D)E_x + N_0Q(D)) = ||p||H(D)Q(D)E_x \quad (4)$$

$$W(D) = \frac{||p||H(D)E_x}{||p||^2Q(D)E_x + N_0} \quad (5)$$

(b)

$$\begin{aligned} S_E(D) &= E[E(D)E^*(D^{-*})] \quad (6) \\ &= E[(H(D)X(D) - W(D)Y(D))(H^*(D^{-*})X^*(D^{-*}) - W^*(D^{-*})Y^*(D^{-*}))] \quad (7) \\ &= H(D)H^*(D^{-*})E_x + W(D)W^*(D^{-*})(||p||^2Q^2(D)E_x + N_0Q(D)) \\ &\quad -W(D)||p||Q(D)H^*(D^{-*})E_x - ||p||W^*(D^{-*})Q(D)H(D)E_x \quad (8) \end{aligned}$$

Substituting the W(D) found in part a, we obtain

$$S_E(D) = \frac{H(D)H^*(D^{-*})N_0E_x}{||p||^2Q(D)E_x + N_0} \quad (9)$$

(c) If H(D)=1, the operation performed becomes a MMSE linear estimation with

$$W(D) = \frac{||p||E_x}{||p||^2 Q(D) E_x + N_0}$$

PROBLEM 2.

(a) For the Zero-forcing equalizer we have,

$$W_{ZFE}(D) = \frac{1}{||p||Q(D)} = \frac{||p||}{a^* D^{-1} + ||p||^2 + aD} \tag{10}$$

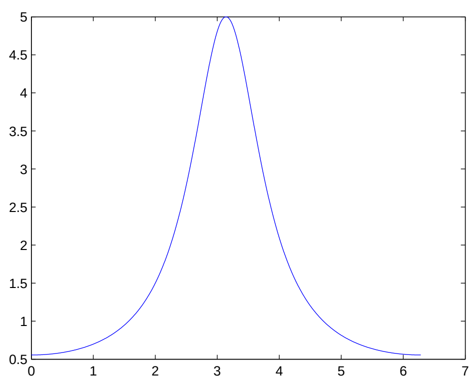
For the MMSE-LE we have calculated in Problem 1- Part (c) that,

$$\begin{aligned} W_{MMSE-LE}(D) &= \frac{||p||E_x}{||p||^2 Q(D) E_x + N_0} && (11) \\ &= \frac{||p||}{||p||^2 Q(D) + \frac{N_0}{E_x}} && (12) \\ &= \frac{||p||}{a^* D^{-1} + aD + ||p||^2 + \frac{N_0}{E_x}} && (13) \\ &= \frac{||p||}{a^* D^{-1} + b + aD} && (14) \end{aligned}$$

Since

$$b = ||p||^2 (1 + \frac{1}{SNR_{MFB}}) = ||p||^2 + \frac{N_0}{E_x}$$

(b) Notice the pole-like behaviour for a=0.9, and difference in power between W_{ZFE} and $W_{MMSE-LE}$



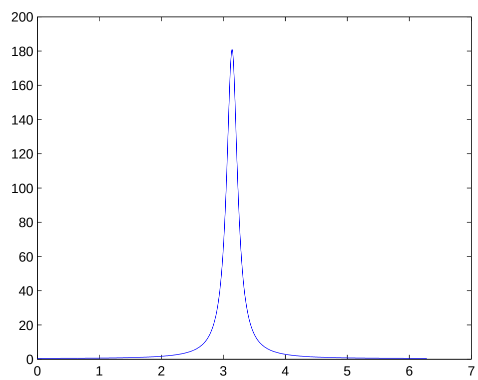


Figure 1: W_{ZFE} for $a = 0.5$ (left) and $a = 0.9$ (right)

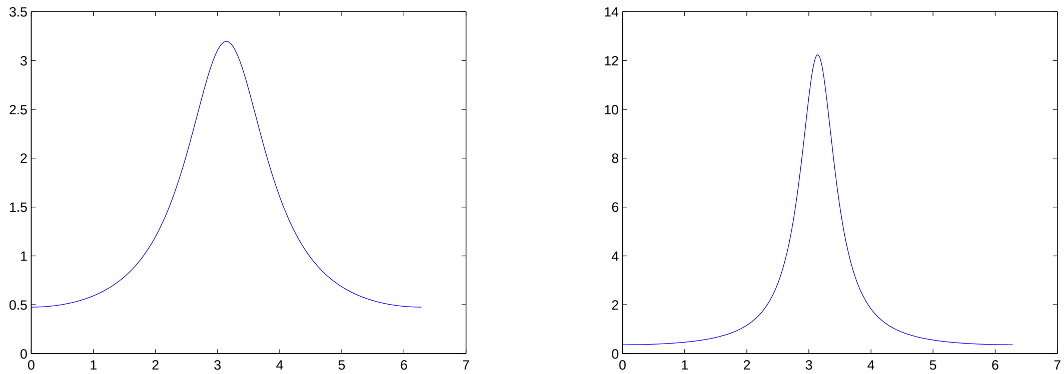


Figure 2: $W_{MMSE-LE}$ for $a = 0.5$ (left) and $a = 0.9$ (right)

(c) The roots are given by:

$$r_1 = \frac{-b - \sqrt{b^2 - 4aa^*}}{2a} \tag{15}$$

$$r_2 = \frac{-b + \sqrt{b^2 - 4aa^*}}{2a} \tag{16}$$

$b^2 - 4aa^*$ is always real since b^2 and aa^* is real.

$$b^2 - 4aa^* = [||p||^2(1 + \frac{1}{SNR_{MFB}})]^2 - 4aa^* \tag{17}$$

$$\geq [||p||^2]^2 - 4aa^* = (1 - aa^*)^2 > 0 \tag{18}$$

(d) $W_{MMSE-LE}(D)$ can be written as:

$$W_{MMSE-LE}(D) = \frac{||p||}{a^* D^{-1} + b + aD}$$

By regrouping and expanding it into partial fractions we obtain:

$$W_{MMSE-LE}(D) = \frac{||p||}{a} \frac{D}{(D - r_1)(D - r_2)} \tag{19}$$

$$= \frac{||p||}{a}[\frac{A}{D - r_1} + \frac{B}{D - r_2}] \tag{20}$$

$$= \frac{||p||}{a(r_1 - r_2)}(\frac{r_1}{D - r_1} - \frac{r_2}{D - r_2}) \tag{21}$$

(e) Taking the inverse D-transform (assuming the sequences are stable) we obtain

$$\frac{r}{D - r} \Longleftrightarrow \begin{cases} (-\frac{1}{r})^n u(n) & , \quad \text{for } |r| > 1 \\ (\frac{1}{r})^n u(-n-1) & , \quad \text{for } |r| < 1 \end{cases}$$

Thus, for the MMSE-LE we have:

$$w_0 = \frac{||p||}{a(r_1 - r_2)} = \frac{||p||}{\sqrt{b^2 - 4aa^*}}$$

For the ZFE, we take $\frac{1}{SNR_{MFB}} = 0$ and thus $b = ||p||^2$ to obtain

$$w_0 = \frac{||p||}{1 - aa^*}$$

(f)

$$SNR_{MFB} = \frac{||p||^2 E_x}{N_0} = \frac{||p||^2}{\sigma^2} = 10||p||^2$$

$$\begin{aligned}
SNR_{ZFE} &= \frac{E_x}{\sigma_{ZFE}^2} = \frac{||p||E_x}{N_0 w_0} && (22)\\
&= 10||p||\frac{1-aa^*}{||p||} = 10(1-aa^*) && (23)\\
\sigma_{ZFE}{}^2 &= \frac{\sigma^2 w_0}{||p||} = \frac{\sigma^2}{(1-aa^*)} && (24)\\
&= \frac{1}{10(1-aa^*)} && (25)\\
\gamma_{ZFE} &= 10\log\frac{||p||^2}{(1-aa^*)} = 10\log\frac{(1+aa^*)}{(1-aa^*)} && (26)
\end{aligned}$$

Similarly,

$$\begin{aligned}
SNR_{MMSE-LE} &= \frac{E_x}{\sigma_{MMSE-LE}{}^2} = \frac{||p||E_x}{N_0 w_0} && (27)\\
&= 10||p||\frac{\sqrt{b^2-4aa^*}}{||p||} = 10\sqrt{b^2-4aa^*)} && (28)\\
\sigma_{MMSE-LE}^2 &= \frac{\sigma^2 w_0}{||p||} = \frac{\sigma^2}{\sqrt{b^2-4aa^*}} && (29)\\
&= \frac{1}{10\sqrt{b^2-4aa^*}} && (30)\\
\gamma_{MMSE-LE} &= 10\log\frac{||p||^2}{\sqrt{b^2-4aa^*}} = 10\log\frac{1+aa^*}{\sqrt{b^2-4aa^*}} && (31)\\
&= 10\log\frac{1+aa^*}{\sqrt{(1+aa^*+0.1)^2-4aa^*}} && (32)
\end{aligned}$$

(g) Using what we have found in part (f)

$$\gamma_{ZFE} = \begin{cases} 10\log 1 = 0 & , \quad a=0 \\ 10\log\frac{1.25}{0.75} = 2.218 & , \quad a=0.5 \\ 10\log\frac{2}{0} = \infty & , \quad a=1 \end{cases}$$

$$\gamma_{MMSE-LE} = \begin{cases} 10\log\frac{1}{\sqrt{1.21}} = -0.41 & , \quad a=0 \\ 10\log\frac{1.25}{0.907} = 1.39 & , \quad a=0.5 \\ 10\log\frac{2}{0.64} = 4.95 & , \quad a=1 \end{cases}$$

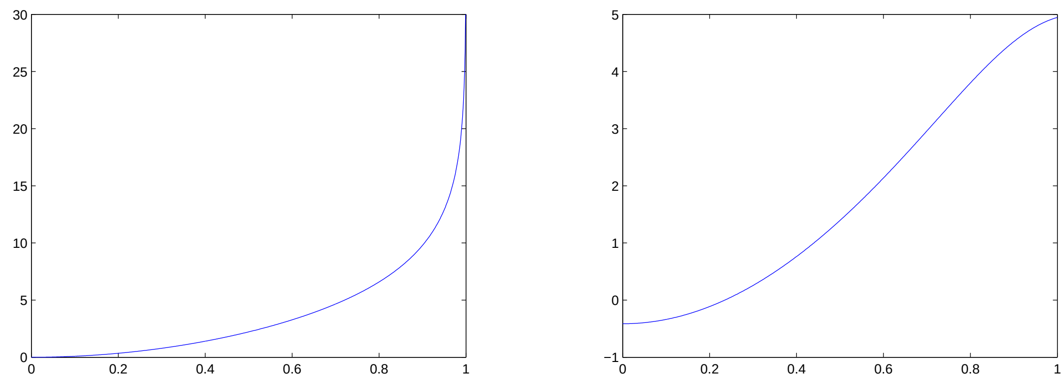


Figure 3: γ_{ZFE} (left) and $\gamma_{MMSE-LE}$ (right)